hyperplane coordinates given by (13), we see that (8) is the condition that $P(X_i \equiv (A_{ii})^{1/2})$ shall lie on the apolar quadratic $\sum b_{ij}x_ix_j = 0$.

Finally we note that if we expand (3) m times, first using the elements of the first column and their cofactors, then the elements of the second column and their cofactors, and in like manner to the last column, and if we then add our results and equate Δ to zero (removing the odd factor m), we get (3) in the form $\sum a_{ij}A_{ij} = 0$, where a_{ij} and A_{ij} are the same as for (8). This shows us that for (1) to be a degenerate quadratic, when m is odd, the point P in (15) must lie on (1). There is no similar simple geometrical description when m is even and (1) is degenerate.

SYRACUSE UNIVERSITY

A CLASS OF UNIVERSAL FUNCTIONS*

BY GORDON PALL

Let a, b, c, d be integers, $a \neq 0$. The function $f(x, y)$ defined by the equation

$$(1) \qquad f(x, y) = axy + bx + cy + d$$

will be called *universal* if $f(x, y)$ represents all integers for integral values of x and y.

THEOREM 1. *A necessary and sufficient condition for* (1) *to be universal is that*

$$(2) \qquad b \equiv \pm 1 \text{ or } c \equiv \pm 1 \pmod{a},$$

or $a = 6$, $b \equiv \pm 3$, $c \equiv \pm 2 \pmod{6}$, *or vice versa for* b *and* c.†

The sufficiency is evident. For, if $b = \pm 1 + Ba$,

$$f(x, -B) = \pm x + d - Bc.$$

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* Presented to the Society, December 28, 1931.

† The writer was led to the exceptional form $6xy+3x+2y$ as in the analysis below, but through an oversight he thought it did not represent 7. The error was, fortunately, pointed out by W. L. G. Williams before this paper went to press.

By replacing x by $y-h$ and y by $y-k$, we can vary the coefficients b and c modulo a. By altering the sign of x or y or both we get a, b, c of like sign. By changing the sign of f we get $a>0$. Finally the constant d may be dropped without affecting the generality. Hence f is reduced to a form in which

$$0 \leqq b \leqq \tfrac{1}{2}a, \; 0 \leqq c \leqq \tfrac{1}{2}a, \; d = 0. \tag{3}$$

The process of reduction does not change the absolute value of either b or c (mod a). Hence the theorem will follow if proved for forms satisfying (3).

The theorem is evident for (3) if $a=1, 2, 3, 4$ or if either b or c is equal to 1 or 0. Hence we may suppose $a \geqq 5$, $b \geqq c \geqq 2$. If $|x| \geqq 2$, $|y| \geqq 2$, then

$$|axy + bx + cy| \geqq 2a$$

by (3). If x or y has the values $-1, 0, 1$ the only values of f which may equal 1 or 2 are $a-b-c$, b, c. Hence $a-b-c=1$, and $c=2$, whence either

$$a = 5, \; b = c = 2; \text{ or } a = 6, \; b = 3, \; c = 2.$$

In the first case f fails to represent 3. In the second case f is evidently universal, since

$$6xy+3x+2y+1=(3x+1)(2y+1),$$

and $3x+1$ represents either 2^h or -2^h.

We consider an extension to functions (1) which are $\geqq 0$ for all integers x, $y \geqq 0$. Evidently these occur if and only if $a>0$, $b \geqq 0$, $c \geqq 0$, $d \geqq 0$. Then $f(x, y)$ represents 0 for integers $\geqq 0$ if and only if $d=0$, and then represents 1 only if b or c is 1. Writing

$$\alpha=a, \; \beta=b, \; \xi=x+1, \; \eta=y+1,$$

we have the following result.

THEOREM 2. *Let α, β denote integers, $\alpha>0$, $\beta \geqq 0$. The only functions of the type* (1) *which represent only positive integers for positive integers ξ and η, and represent all such integers for such ξ and η, are*

$$\phi(\xi, \eta) = \xi\eta + (\alpha - 1)(\xi - 1)(\eta - 1) + (\beta - 1)(\xi - 1), \tag{4}$$

and $\phi(\eta, \xi)$.

The function obtained from (4) with $\beta=1$, namely

$$\psi(x, y) \equiv xy + (\alpha - 1)(x - 1)(y - 1), \tag{5}$$

is of special interest in view of the property

$$\psi(\psi(x, y), z) = \psi(x, \psi(y, z)). \tag{6}$$

If we call $\psi(x, y)$ the α-product of x and y and denote it alternatively by $x \circ y$, then

$$x \circ y = y \circ x, \quad (x \circ y) \circ z = x \circ (y \circ z),$$

and the positive integers may be studied under α-multiplication. We have $1 \circ y = y$, so that 1 acts as identity element. We call the positive integer $y > 1$ an α-prime if its only divisors under α-multiplication are 1 and y. For example, 2, 3, 4, 6, 7, 9, 10, 12, 15, 16, 19, $\cdots$ are 2-primes; but 5, 8, 11, 13, 14, 17, 18, $\cdots$ are 2-composite since, if $\alpha = 2$, $5 = 2 \circ 2$, $8 = 2 \circ 3$, $\cdots$, $13 = 3 \circ 3$, $\cdots$. For any α there are infinitely many α-primes.

It is easy to see that α-decomposition (apart from order) into α-primes is unique if and only if $\alpha = 1, 2$. That it is unique if $\alpha = 2$ is plain from the equivalence of the equations

$$n = 2xy - x - y + 1, \quad 2n - 1 = (2x - 1)(2y - 1).$$

It follows also from the equivalence that p is a 2-prime if and only if $2p - 1$ is an ordinary prime.

If $\alpha = 3$, however, $4 \circ 8 = 2 \circ 19$ but 2, 4, 8 and 19 are distinct 3-primes. It is easy to construct the ideal divisors, restoring unique factorization. Generally, the theory is equivalent to that of the set of numbers $\alpha x + 1$, $x = 0, 1, 2, \cdots$, under ordinary multiplication.

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