QUADRATIC PARTITIONS—I

BY E. T. BELL

1. *Introduction.* This is the preliminary and longest note of a series which, by the kindness of the editors of this Bulletin, I hope to publish from time to time, giving some of the numerous general arithmetical theorems of a particular type which I have been accumulating for several years. To make this series self contained, I first recall the necessary definitions, and give once for all a few formulas that will be used repeatedly. Results and methods of two previous papers are indicated by numbered references.*

Subsequent notes will contain only theorems, with statements of the elementary identities from which they follow. This will be sufficient to enable anyone who wishes to retrace the details of the proofs and verify the conclusions. I believe that present conditions of mathematical publication in this country demand the utmost brevity consistent with reasonable clarity.

2. *Parity.* Let $\xi=(x_1, \cdots, x_n)$ be a one-row matrix or vector in which the *elements* $x_1, \cdots, x_n$ are in a given field K. Write $-\xi\equiv(-x_1, \cdots, -x_n)$. If $f(\xi)$ is a single finite real or complex number whenever $x_1, \cdots, x_n$ are in K, we say that $f(\xi)$ is *uniform over* K. Let $f(\xi)$ be uniform over K. Then, if $f(-\xi)=f(\xi)$, we say that $f(\xi)$ has *parity* $p(n|)$ in ξ; if $f(-\xi)=-f(\xi)$, and if further $f(0, \cdots, 0)=0$, the parity is $p(|n)$. Let us denote by $\xi_i, \eta_j, (i=1, \cdots, r; j=1, \cdots, s)$, vectors in K, having no element in common. Then, if $f(\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s)$ has parity $p(n_i'|)$ in ξ_i, and parity $p(|n_j'')$ in η_j $(i=1, \cdots, r; j=1, \cdots, s)$, we shall agree to say that $f(\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s)$ has parity $p(n_1', \cdots, n_r'|n_1'', \cdots, n_s'')$ in $(\xi_1, \cdots, \xi_r|\eta_1, \cdots, \eta_s)$, and we write

* (1) *Arithmetical paraphrases*, Transactions of this Society, vol. 22 (1921), pp. 1–30; 198–219; (2) *A revision of the Bernoullian and Eulerian functions*, this Bulletin, vol. 28 (1922), pp. 443–450. The material in (1) is included and generalized in (3) *Algebraic Arithmetic*, American Mathematical Society Colloquium Publications, vol. 7, 1927, Chapters 2, 3; ibid., pp. 146–159, contain a complete account of the umbral calculus used in (2) and in some of the present notes.

$$f(\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s) \equiv f(\xi_1, \cdots, \xi_r \mid \eta_1, \cdots, \eta_s).$$

The field K will be given explicitly, or by the context. Unless otherwise noted, K is the field of all rational numbers, and the values of the elements of $\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_s$ are rational integers.

3. *Notation.* As in the references in §1, $n, n_i, d, \delta, d_i, \delta_i, t, t_i, m, \tau, m_i, \tau_i, \nu, \nu_i, a, b, \mu, \mu_i, (i=1, \cdots)$, denote integers, of which $n, n_i, d, \delta, d_i, \delta_i, t, t_i$ are greater than zero, and otherwise unrestricted, m, τ, m_i, τ_i are greater than zero and odd, ν, ν_i, a, b are greater than, equal to, or less than zero and are unrestricted, μ, μ_i are greater than or less than zero and odd.

If one or more of $n, \cdots, \mu_i$ occur under $\sum$, the sum refers to all $n, \cdots, \mu_i$ as defined.

A sum $\sum_a^b$ in which $b<a$ is vacuous, and is to be suppressed.

The *umbra* (see §1, references (2), (3)) of the sequence $\xi_0, \xi_1, \cdots, \xi_s, \cdots$, in which the first element has the suffix zero, is ξ. Symbolically, $\xi^s \equiv \xi_s, (s=0, 1, \cdots)$. I define the (umbral) *indefinite integral* of ξ to be ξ', where ξ' is the umbra of $\xi_{s+1}/(s+1), (s=0, 1, \cdots)$. The even suffix notation is used (as in paper (2)) for the numbers of Bernoulli, Euler, Genocchi, and Lucas, whose respective umbrae are B, E, G, R. Hence B', E', G', R' are defined. The sequences of functions associated with B, G, E, R, whose respective umbrae are β, γ, η, ρ, are as in the paper (2). If necessary to indicate the argument x, we shall write $\beta(x)$, etc. Thus $\beta(x)$ is the umbra of $\beta_s(x), (s=0, 1, \cdots)$; $\beta'(x)$ is the umbra of $\beta_{s+1}(x)/(s+1), (s=0, 1, \cdots)$.

4. *Partitions.* Let n (§3) be constant, and let $Q(x_1, \cdots, x_p)$ be any polynomial in $x_1, \cdots, x_p$ with coefficients in K(§2). The totality of vectors $(x_1, \cdots, x_p)$, whose elements are in K, such that $n=Q(x_1, \cdots, x_p)$, will be called the *Q-partition* of n. If this partition contains an infinity of distinct vectors, we impose conditions $C(x_1, \cdots, x_p)$ upon $x_1, \cdots, x_p$ such that, subject to $C(x_1, \cdots, x_p)$, the Q-partition contains only a finite number of distinct vectors, and refer to this as a *restricted* partition. Restrictions will always be stated explicitly; otherwise, the partition is unrestricted. If $x_1, \cdots, x_p$ occur under $\sum$, the sum is with respect to the partition, and the limits need not be otherwise indicated.

If Q above is homogeneous of degree 2, the partition is called *quadratic.*

5. *Special Functions*. The following will occur frequently. If x is real and positive, $[x]$ in an exponent or as a summation limit denotes the greatest integer in x. If y is real and different from zero, sgn y is defined (as usual) by sgn $y=y^{-1}|y|$, and sgn $0=0$. Hence, for real u, v,

$$\sin (u \operatorname{sgn} v) = \operatorname{sgn} v \sin u;$$

for z real, $\neq 0$, and x, y real,

$$\cos (x \operatorname{sgn} z) = \cos x,$$

$$\operatorname{sgn} z \cos x \cos y \pm \sin x \sin y = \operatorname{sgn} z \cos (x \mp y \operatorname{sgn} z),$$

$$\operatorname{sgn} z \sin x \cos y \pm \cos x \sin y = \operatorname{sgn} z \sin (x \pm y \operatorname{sgn} z).$$

Referring to §3, we define $e(\nu)$ to be $+1$ if ν is even; -1 if ν is odd. Refer to §2 for ξ. If $f(\xi)$ has an expansion of the form

$$\sum A_{a_1,\cdots,a_n} x_1^{a_1} \cdots x_n^{a_n},$$

which is convergent in some non-zero region of the n-space of ξ, we say that $f(\xi)$ is an *entire function of* ξ. In particular, a polynomial in $x_1, \cdots, x_n$ is entire in ξ.

The previous notation (paper (1), p. 207) $\phi_{abc}(x, y)$ for the doubly periodic functions of the second kind,

$$\phi_{abc}(x, y, q) \equiv \phi_{abc}(x, y) = \vartheta_1' \vartheta_a(x + y)/(\vartheta_b(x)\vartheta_c(y)),$$

of which there are 16, will be used. The remaining 48 expansions, not available in previous work, have been obtained by D. A. F. Robinson, and will be printed elsewhere.*

6. *Special Umbral Identities*. In the passage from trigonometric identities to their equivalents in terms of parity functions, the trigonometric terms having simple poles at the origin play a particular part; see paper (1), p. 204. Such terms contribute sums of parity functions one or more of whose arguments are in arithmetical progression. The residue of the pole must be zero in any trigonometric identity paraphrased. If it is not immediately obvious that the residue vanishes, the fact that it must gives a subsidiary theorem. The following formulas, which will be frequently used, enable us to write down the residues without calculations in one type of theorem; ξ is umbral as in §3.

* Probably in the Transactions of the Royal Society of Canada. These expansions will be stated when used.

$$2 \operatorname{ctn} x \sin (\xi x + y) = 2x^{-1}\xi_0 \sin y + \cos \{\beta'(\xi)x + y\},$$
$$2 \operatorname{ctn} x \cos (\xi x + y) = 2x^{-1}\xi_0 \cos y - \sin \{\beta'(\xi)x + y\};$$

$$4 \tan x \sin (\xi x + y) = \cos \{\gamma'(\xi)x + y\},$$
$$4 \tan x \cos (\xi x + y) = - \sin \{\gamma'(\xi)x + y\};$$

$$2 \sec x \sin (\xi x + y) = \sin \{\eta(\xi)x + y\},$$
$$2 \sec x \cos (\xi x + y) = \cos \{\eta(\xi)x + y\};$$

$$\csc x \sin (\xi x + y) = x^{-1}\xi_0 \sin y + \cos \{\rho'(\xi)x + y\},$$
$$\csc x \cos (\xi x + y) = x^{-1}\xi_0 \cos y - \sin \{\rho'(\xi)x + y\}.$$

7. *Trigonometric Identities.* From the identities on pages 204–5 of paper (1), we write down eight which generalize them and greatly reduce algebraic work later. Refer to §§3, 5, and write

$$N \equiv [\tfrac{1}{2} | \nu |], \quad M \equiv [\tfrac{1}{2}(| \nu | - 1)].$$

Then

$$\csc x \sin (\nu x + y) = [\{1 - e(\nu)\} \operatorname{ctn} x + e(\nu) \csc x] \sin y$$
$$+ \operatorname{sgn} \nu [\{1 - e(\nu)\} \cos y + 2 \sum_{r=1}^{N} \cos \{(2r - e(\nu)) x \operatorname{sgn} \nu + y\}];$$

$$(-1)^N \sec x \sin (\nu x + y) = e(\nu) \sec x \sin y$$
$$+ \{1 - e(\nu)\} \operatorname{sgn} \nu \tan x \cos y + \{1 - e(\nu)\} \sin y$$
$$+ 2 \sum_{r=1}^{N} (-1)^r \sin \{(2r - e(\nu)) x \operatorname{sgn} \nu + y\};$$

$$\tan x \sin (\nu x + y) = (-1)^N e(\nu) \tan x \sin y$$
$$+ (-1)^M \{1 - e(\nu)\} \operatorname{sgn} \nu \sec x \cos y (-1)^M$$
$$+ (-1)^M \operatorname{sgn} \nu [e(\nu) \cos y - (-1)^M \cos (\nu x + y)$$
$$+ 2 \sum_{r=1}^{M} (-1)^r \cos \{(2r - 1 + e(\nu)) x \operatorname{sgn} \nu + y\}];$$

$$\operatorname{ctn} x \sin (\nu x + y) = \Big[e(\nu) \operatorname{ctn} x + \{1 - e(\nu)\} \csc x\Big] \sin y + \operatorname{sgn} \nu \Big[e(\nu) \cos y + \cos (\nu x + y) + 2 \sum_{r=1}^{M} \cos \{(2r - 1 + e(\nu)) x \operatorname{sgn} \nu + y\} \Big].$$

The remaining four are written down from these by replacing y by $y+\pi/2$. All will be used to reduce terms involving csc, sec, tan, ctn in trigonometric identities before passing to parity functions.

8. *General Umbral Identities.* The principle of paraphrase stated in paper (1), pages 4, 5, can be extended to umbral sines and cosines, identities between which paraphrase into identities between *entire* functions as defined in §5. That is, the *elements* of the one-row matrices, or vectors, in the principle as previously stated, can be replaced by umbrae. It is necessary only to define parity for functions of umbrae, and it will be sufficient to state the definitions for functions of one umbra ξ. If $f(x) \equiv f(x\,|)$ is an entire function of the ordinary x, we say that $f(\xi)$ $(\equiv f(\xi\,|))$ has *parity* $p(1\,|)$ in ξ. According to this definition and what precedes, $f(\xi)$ is of the form

$$p_0\xi_0 + p_2\xi_2 + \cdots + p_{2s}\xi_{2s} + \cdots,$$

where the series either converges or terminates. If $g(-x) = -g(x)$, we say that $g(\xi)$ $(\equiv g(\,|\xi))$ has parity $p(\,|1)$ in ξ, and $g(\xi)$ is of the form

$$p_1\xi_1 + p_3\xi_3 + \cdots + p_{2s+1}\xi_{2s+1} + \cdots.$$

The condition $g(0)=0$ is not imposed, as it is not required here.

To see how the principle goes over to umbrae the following case will suffice. Let

$$f(\xi) = p_0\xi_0 + p_2\xi_2 + \cdots + p_{2s}\xi_{2s};$$

let x be an ordinary umbra, and $a, b, \cdots, c$ umbrae such that

$$\cos ax + \cos bx + \cdots + \cos cx \equiv 0$$

is an identity in x. Then

$$f(a) + f(b) + \cdots + f(c) = 0.$$

For, the given identity implies

$$a_{2r} + b_{2r} + \cdots + c_{2r} = 0, \quad (r = 0, 1, \cdots);$$

and therefore

$$p_0(a_0 + b_0 + \cdots + c_0) + p_2(a_2 + b_2 + \cdots + c_2) + \cdots + p_{2s}(a_{2s} + b_{2s} + \cdots + c_{2s}) = 0;$$

which is the stated conclusion.

CALIFORNIA INSTITUTE OF TECHNOLOGY

ON SYMMETRIC PRODUCTS OF TOPOLOGICAL SPACES*

BY KAROL BORSUK AND STANISLAW ULAM

1. *Introduction.* This paper is devoted to an operation that is defined for an arbitrary topological† space E and is analogous to the operation of constructing the combinatorial product spaces.‡ We shall be concerned with the topological properties of point sets defined by means of the above operation when executed on the segment $0 \leqq x \leqq 1$.

Let E be an arbitrary topological space. Let E^n denote the nth topological product of the space E, that is, the space whose elements are ordered systems $(x_1, x_2, \cdots, x_n)$ of points $x_i \epsilon E$. By a *neighborhood* of a point $(x_1, x_2, \cdots, x_n)$, we understand the set of all systems $(x_1', x_2', \cdots, x_n')$, where x_i' belongs to a neighborhood u_i of the point x_i in the space E.‡

The operation with which we are concerned in this paper consists in constructing a space which we shall call the nth *symmetric product* of the space E and denote by $E(n)$. Its elements are *non-ordered* systems of n points (which may be different or not) belonging to E. Two systems differing only by the order or multiplicity of elements are considered identical. A non-ordered system or simply a *set* consisting of n points $x_1, \cdots, x_n$ from the space E will be denoted by $\{x_1, x_2, \cdots, x_n\}$. If u_i is a neighborhood of the point x_i in the space E, then the set of all systems

* The definition of *symmetric products* is given below.

† In the sense of Hausdorff, *Grundzüge der Mengenlehre*, p. 228.

‡ See, for example, F. Hausdorff, *Grundzüge der Mengenlehre*, p. 102.