WEATHERBURN'S SECOND VOLUME

Differential Geometry of Three Dimensions. Volume II. By C. E. Weatherburn. Cambridge University Press, 1930. xii+239 pp.

The first volume of this work (see this Bulletin, vol. 34 (1928), pp. 785–786) covered the fundamentals of metric differential geometry of curves and surfaces. The present volume, though containing certain classical material supplementing that of the first volume, is primarily devoted to a consequential exposition of the author's published contributions to the subject.

The treatment in both volumes is in terms of vectors. But, whereas the first volume employs, except in the last chapter, on differential invariants, only the algebra of vectors, the second volume uses also, and to a great extent, the differential and integral calculus of vectors. Moreover, dyadics are introduced in the middle of the volume and are used to good effect throughout the later chapters.

Chapters 2, 3, and 8 contain the author's work on families of curves on a surface. In Chapters 5 and 6 the calculus of vectors referred to curvilinear coordinates in space is developed and applied to families of surfaces. Chapters 10, 11, and 12 have to do with transformations, small deformations, and applicability of surfaces. In Chapter 13 are found the author's contributions to the theory of curvilinear congruences, and in Chapter 9 those bearing on the parallelism of Levi-Civita.

In treating the family of curves $\phi(u, v) = \text{const.}$ on a surface, the author makes extensive use of the function $\psi = 1/(\Delta_1\phi)^{1/2}$. Since the distance between the curve $\phi = \phi_0$ and the curve $\phi = \phi_0 + \Delta\phi$, measured along the orthogonal trajectories of the curves $\phi = \text{const.}$, is approximately $\psi\Delta\phi$, he calls ψ the distance function for the given family of curves and the curves $\psi = \text{const.}$ the lines of equidistance for this family. Similarly, in treating a family of surfaces in space $\phi(u, v, w) = \text{const.}$, he introduces the distance function $\psi = 1/(\Delta_1\phi)^{1/2}$, the surfaces of equidistance $\psi = \text{const.}$, and the lines of equidistance $\psi = \text{const.}$, $\phi = \text{const.}$

The distance function enables the author to simplify greatly the formal work. His skilful use of it in connection with the study of ruled surfaces in Chapter 4 is particularly to be commended. There is, however, an important point in connection with it which he fails to mention. Though the function itself is invariantly connected with the given function ϕ, the lines of equidistance for a given family of curves on a surface and the surfaces of equidistance for a given family of surfaces are not fixed respectively by the family of curves and the family of surfaces; they vary with the choice of the function ϕ employed to define the given family. On the other hand, the lines of equidistance for a family of surfaces are uniquely determined by the family itself.

In applying dyadics to a family of curves on a surface, the author introduces what he calls the tendency, the moment, and the swerve of the family in a given direction. If $\boldsymbol{t}$ is the unit vector tangent to the curve of the family passing through a point P and $\boldsymbol{a}$ is the unit vector in the given direction at P, these quantities are respectively $\boldsymbol{a}\cdot\nabla \boldsymbol{t}\cdot\boldsymbol{a}$, $\boldsymbol{a}\cdot(\nabla \boldsymbol{t}\times\boldsymbol{t})\cdot\boldsymbol{a}$, and $\boldsymbol{a}\cdot(\nabla \boldsymbol{t}\times\boldsymbol{n})\cdot\boldsymbol{a}$, where $\boldsymbol{n}$ is the unit vector normal to the surface at P. More familiar expressions for, say, the first and the third are $\sin\omega(\gamma\cos\omega+\gamma'\sin\omega)$ and $\cos\omega\,(\gamma\cos\omega+\gamma'\sin\omega)$, where ω is the angle from $\boldsymbol{t}$ to $\boldsymbol{a}$ and γ and γ' are respectively the geodesic curvatures of the given curves and their orthogonal trajectories.

It would appear that these concepts were motivated more by the vector analysis than by the geometry of the situation. On the other hand, similar concepts are employed, in the study of curvilinear congruences, to yield useful and striking geometric results.

Levi-Civita's concept of parallel displacement is set forth in excellent fashion except for the omission of the definition, in terms of it, of geodesic curvature. An interesting application consists in the theorem to the effect that a system of curves on a surface clothes the surface in the sense of Tchebychef if and only if the unit vectors tangent to the curves of each family are parallel with respect to the curves of the other family.

In the map of a surface S on a surface S' parameters u, v common to the two surfaces are chosen so that corresponding points have the same coordinates (u, v). The discussion is restricted to maps for which the components dx', dy', dz' of the vector element $d\boldsymbol{r}'$ on S' are linear homogeneous functions of the components dx, dy, dz of the corresponding vector element $d\boldsymbol{r}$ on S, or, what is the same thing, to maps for which $d\boldsymbol{r}'=\Phi\cdot d\boldsymbol{r}$, where Φ is a dyadic depending on u, v. The map is thus made to depend on a dyadic and the study of it is in terms of this dyadic.

The small deformations of a surface discussed by the author are perfectly general; so-called infinitesimal deformations, which preserve the linear element except for terms of the second and higher orders, are considered briefly as a special case. The treatment of applicability of surfaces is along classical lines.

The greater part of the references in the book are to the author's published papers. While this is quite proper, inasmuch as the author's contributions are largely novel, there are regrettable oversights of original sources in the case of a few well known theorems.

The tendency of the vector analysis to override the geometry is greater in this than in the previous volume. Only the author's strong interest in geometry and keen geometrical insight save the book from becoming too formal. As an extension of the first volume and an exposition of the author's widely published researches, it should make a strong appeal both to the student and to the specialist.

W. C. GRAUSTEIN