

vol. 25 (1902), pp. 59–84, 85–96, 261–268; vol. 27 (1905), pp. 77–102, 173–188, the present reviewer established the existence and derived the equations of about two hundred sextic ruled surfaces, including all those in the book under review, and about eighty others which under Mr. Edge's basis of classification should be counted as distinct types. No criterion was found, to establish the completeness; thus far, no other forms have been published.

Of the specific omissions, the following have distinct double curves:

two double cubics and two double conics	$p=0$;
four double conics+two double generators	$p=0$;
double cubic and three double conics	$p=1$;
double quartic, two double conics and a double generator	$p=1$;
double quartic, double conic, double directrix and two double generators	$p=1$.

Then there are fourteen other forms without rectilinear directrices, having tacnodal or oscnodal curves. Thus, of the four double conics, two may approach coincidence, forming a tacnodal conic, or three may approach coincidence, forming an oscnodal conic. Of the large number of omissions of those having a directrix line which may or may not be a generator, a frequent sample is that caused by a compound involution.

For ruled surfaces having a directrix line, the reviewer is inclined to feel that the methods of Wiman and of Sisam, *American Journal of Mathematics* (vol. 29 (1907), pp. 48–100), are at least as powerful and as comprehensive as those developed by the author. Since all these papers are featured in the *Encyklopädie* (III C 8; Art. 52), the author should at least have mentioned them. There, if the reviewer has established his reasons for protesting the content of one paragraph, he now wishes to emphasize that Mr. Edge has produced an excellent book that will be of very great value in the study of various branches of algebraic geometry.

VIRGIL SNYDER

HITHERTO UNPUBLISHED TREATISE OF STEINER

Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln worunter eine grosse Anzahl neuer Untersuchungen und Sätze vorkommen in einem systematischen Entwicklungsgange dargestellt. By Jakob Steiner. Edited by Rud. Fueter and F. Gonseth. Zürich and Leipzig, Orell Füssli, 1931. 8vo, xviii+345 pp. Price in marks: paper, 10.80; cloth 12.80.

This is an original manuscript written by Steiner more than a century ago. It now appears in print for the first time. It is published under the auspices of the Swiss Naturalist Society with the assistance of the Escher-Abegg Foundation for Scientific Research at the University of Zürich. It appears as volume five in a series of publications of the Swiss Mathematical Society.

Jakob Steiner (1796–1863), the great Swiss geometer, wrote this treatise on the circle and the sphere probably during the years 1823–26 while a private teacher in Berlin. The manuscript consists of 360 carefully written pages with title, book and chapter headings, evidently all prepared for immediate publication. For some reason, however, it was never published. The editors suggest

that it may have been intended for the geometrical encyclopaedia planned at that time by Steiner, but never realized.

An earlier manuscript, probably written in 1825, whose table of contents is complete, was to contain three books, the first treating the circle, the second, the sphere, and the third, circles on the sphere. Of this work, only the first two chapters and a part of the third in the first book were written up. This manuscript was evidently superseded by the one now published, which, however, does not contain a discussion of circles on the sphere.

Steiner dated many of his manuscripts, but only one date, Oct. 1, 1825 at the beginning of book three, occurs in the completed manuscript. This date and certain references by Steiner in published memoirs indicate that this work was finished in 1826.

The question naturally arises: Why was this important treatise, painstakingly prepared for publication by Steiner himself, not included in Steiner's *Gesammelte Werke* (1881–82), edited by Weierstrass with the assistance of Geiser?

In the preface of the second volume of the *Gesammelte Werke*, Weierstrass writes: “. . . The expectation shared by many that this volume would contain a series of interesting, hitherto unpublished articles from the Steiner manuscript legacy (Nachlass) rests on a false impression of the contents of this legacy. Professor Geiser of Zürich has had the goodness to inform me that it consists chiefly of works in preparation and various drafts of a number of memoirs already published and that they would require thorough revision to be of use”

What happened to most of Steiner's manuscript legacy is told by Bützberger:* “Long after the death of Steiner, this manuscript was found on the floor of the Library of the Naturalist Society of Berne where it had been left in a box, disarranged, forgotten and in part already decomposed. Professor Graf found it there and turned it over to the author to put it in order and turn it to good account.”

The title given by Bützberger a few lines further on is evidently that of the older manuscript which was never finished. He states that numerous other Steiner manuscripts were found in this box. He does not give the date of this discovery, but it had evidently happened only a short time before—probably at least thirty years after the death of Steiner and twelve years after the publication of his collected works.

Bützberger had ten volumes of Steiner's lecture notes, letters, etc. bound and placed in the Library of the University of Berne. He kept the more important manuscripts and published results from some of them later,† but died before he had completed the work of editing the Steiner manuscripts.

* F. Bützberger, *Zum 100. Geburtstage Jakob Steiners*, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, vol. 27 (1896), p. 162, beginning with line 24.

† F. Bützberger, *Ueber Bizenrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion*, Leipzig, Teubner, 1913. See also review by Emch, this Bulletin, vol. 20 (1913–14), pp. 412–415.

In the summer of 1928, Arnold Emch* examined the completed manuscript on the circle and sphere, then in possession of Mrs. Bützberger at Zürich. Very probably its present appearance is due, at least in part, to his efforts to interest the Swiss Mathematical Society in its publication.

The treatise is divided into four books, each containing two chapters further subdivided into sections numbered continuously through the entire volume.

The first book deals with centers, lines of similitude and planes of similitude of circles and of spheres; the second with powers and loci of equal powers of circles and of spheres; the third with common powers of circles and spheres and the fourth with systems of circles and systems of spheres that intersect at given constant angles. In each of the books, the first chapter deals with certain properties of circles and the second with the corresponding properties of spheres. In each chapter, theorems are stated and proved, followed by problems and figures illustrating these properties.

The treatment in the fourth book is very general and complete. Here are found the most important contributions of the work—properties rediscovered by mathematicians many years later and often in a more restricted form, for example, orthogonal systems of circles and spheres instead of systems intersecting at any given angle. Here Steiner also develops the properties of surfaces that are envelopes of systems of spheres. In particular, the surface discussed on pages 261–267 is the Dupin cyclide.

One of the great advances in method instituted by Steiner was to treat contact as a special case of intersection; another was a happy system of notation that reduced complicated problems to simple form. Both of these methods are made use of freely in the present volume.

Highly important as this treatise is in content and method, it is fully as valuable as a complete and perfected production of the great geometer, revealing the clearness, simplicity and rigor of his style. The editors have taken great care to preserve this work as a classic and have made no attempt to introduce changes where, in the light of our present knowledge, improvements could be made. Annotations have been inserted by the editors only where considered absolutely necessary, "small additions that Steiner himself would have made for the press in his lifetime," and these are enclosed in parentheses.

Some of the figures, merely sketched by Steiner, have been redrawn more carefully, but the only one completely new is Fig. 60, the large diagram on the extension of the last page. Even in this figure, the basic circles and lines were drawn by Steiner (Fig. 48). The diagram was completed by J. Züllig from Steiner's description of it.

The excellent typography, paper and binding of the volume are in keeping with the worthy treatise to which they lend added years and a wide sphere of usefulness. The editors, the publishers and all directly or indirectly concerned in the preservation and publication of this manuscript richly deserve the deepest gratitude of every mathematician. In reality, after a half century, another volume has just been added to Steiner's *Gesammelte Werke*.

T. R. HOLLCROFT

* A. Emch, *Unpublished Steiner manuscripts*, American Mathematical Monthly, vol. 36 (1929), pp. 273–275.