A RATIONAL QUINTIC SURFACE HAVING NO DOUBLE CURVE

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Montesano in his extensive researches on rational quintic surfaces has noted briefly one such surface obtained by applying a quadratic transformation to the quartic surface known as Nöther's third type.* This quintic surface can be obtained by transformation of a rational sextic surface, due also to Montesano. The plane representation of the quintic which results is the same as that given by Montesano; but, aside from the new method of derivation, there are certain features of the surface and the plane representation not mentioned by him, which it is the purpose of this note to consider.

We start with the rational sextic surface that has a quadruple point Q at which are concurrent 3 coplanar double lines and a triple line, which of course does not lie in the plane of the double lines. If we apply a quadratic transformation of the first type whose fixed conic is the triple line and one of the double lines, and whose fixed point is on another of the double lines, we obtain a rational quintic, noted by Montesano,† which has two consecutive skew double lines, and which I have described in a previous paper.‡ If, however, we apply to the sextic a quadratic transformation whose fixed conic consists of two of the double lines and whose fixed point is on the triple line we obtain another rational quintic which is the subject of this paper.

The plane system of the rational sextic surface is the web of curves of order 9 having in common 8 triple points $A_1, \cdots, A_8$ and 3 simple points B_1, B_2, B_3. This set of curves may be designated as the web of nonics $8A^3B_1B_2B_3$. The image of the triple line k is the plane sextic $8A^2B_1B_2B_3$. The image of a double line d_i is the cubic $8AB_i$. To B_i corresponds the residual line on the sextic surface in the plane of k and d_i. The tangent cone to the sextic at the quadruple point Q consists of the 4 planes kd_1, kd_2, kd_3, and $d_1d_2d_3$.

* Rendiconti di Napoli, vol. 40 (1901), p. 100.

† Ibid., vol. 46 (1907), p. 66.

‡ This Bulletin, vol. 34 (1928), p. 761.

To the elements of direction on the surface at Q in the plane kd_i correspond the elements of direction at B_i' in the image plane. The point B_i' lies on the cubic $8AB_i$ and is the conjugate of B_i in I_{17}, the involution determined by the sextics $8A^2$. All sextics of this web that pass through a general point of the plane pass also through another point determined by the first. To the surface elements at Q in the plane of the double lines correspond the elements about A_0, the ninth base point of the pencil of cubics of $8A$. If we now apply a quadratic transformation whose fixed conic is d_1d_3 and whose fixed point is a general point O on the triple line k, the resulting surface is of order 5. For it loses the planes kd_1 and kd_3 each twice, and the plane $d_1d_2d_3$ three times. A general ray through Q meets the sextic twice elsewhere. Its transform is another ray through Q. Therefore the quintic has a triple point at Q. Similarly a general ray through O meets the sextic in 3 other points. Such a ray is invariant as a whole; and hence the quintic has a double point at O. The image of the curve of intersection of a general quadric with the sextic is a curve of order 18, $8A^6B_1^2B_2^2B_3^2$. Since the homoloidal quadrics contain d_1 and d_3 and O on the triple line k, the image of the intersection of such a quadric with the sextic is of order 12, and has quadruple points at $A_1, \cdots, A_8$, a double point at B_2, and simple points at B_1 and B_3 and at E_1, E_2, E_3, the three points on the sextic image of the triple line that correspond to O. But this curve of order 12 fulfills 2 more conditions in that it is tangent at B_2' to the cubic $8AB_2B_2'$. For a homoloidal quadric has d_1 and d_3 for generators, and hence its intersection with the sextic surface is tangent to d_2 at Q. Therefore the plane sections of the quintic surface have as images the web of duodecimics $8A^4B_2^2B_2'B_2''B_1B_3E_1E_2E_3$. They may be called the base curves. The point B_2'' is consecutive to B_2' on the cubic $8AB_2B_2'$. The 3 tangent planes to the sextic at O give 2 lines, e_1, e_2, e_3 through Q in the plane $d_1d_2d_3$. They correspond to E_1, E_2, E_3 in the new plane system. The residual lines on the sextic in the planes kd_1 and kd_3 give d_1 and d_3 as simple lines on the quintic. They correspond to B_1 and B_3 in the plane system of the latter. The trace of the quintic in the plane of d_1d_3 is therefore the 5 simple lines d_1, d_3, e_1, e_2, e_3 through Q. We shall call this plane σ. The residual line on the sextic in the plane of the triple line and d_2 gives a conic on the quintic in the same plane, passing through O and Q

and tangent to the plane σ at Q. The image of this conic is B_2. Its plane π is tangent to the quintic along k, the residual intersection being the conic just mentioned. A general plane of the pencil k meets the sextic surface in a residual cubic which has the intersection of its plane with σ for inflectional tangent at Q. By the transformation this cubic becomes a quartic in the same plane, tangent to k at O and having a tacnode at Q, the branches being tangent to σ. Therefore the line k, triple on the sextic, is a simple line of the quintic, joining the nodes Q and O. Also the tangent cone to the quintic at Q is the plane σ taken twice and the plane π. At O the tangent cone is π taken twice. The image of the triple point Q of the quintic surface is the same plane sextic that corresponds to the triple line of the sextic surface, that is, the sextic $8A^2B_2B_2'B_1B_3E_1E_2E_3$. The last 3 points are the residual intersections of the sextic with a net of nonics $8A^3B_1B_2B_3E_1(E_2E_3)$. To this net of nonics corresponded on the sextic surface the plane sections through O. Since by the quadratic transformation such a plane is invariant as a whole, the nonics of this net are the images of the plane sections of the quintic surface by planes through O. Such a plane cuts the sextic in a plane curve having a triple point at O, and double points on d_1, d_2, d_3. Considering the quadratic transformation imposed in such a plane by the general transformation we see that the plane section of the sextic surface becomes a plane quintic with a tacnode at O. The unode at O is therefore a tacnode of the quintic surface. Its image is the cubic $8AB_2B_2'$.

Among the ∞^3 homoloidal quadrics which effect the transformation are the ∞^2 cones whose vertex is Q and which contain the lines d_1, d_3, and k. These cones intersect the sextic surface in skew quintic curves whose images are the sextics of the net $8A^2B_2B_2'$. The cones become planes through Q. Therefore the sextics of the net just mentioned are the images of sections of the quintic surface by planes through its triple point Q. To sections by planes through d_1 correspond the ∞^1 sextics of the net that pass through B_1. Similarly for sections through d_3, e_1, e_2, e_3. The section of the quintic by a general plane through Q has at Q a triple point composed of three linear branches two of which have contact, the tangent lying in σ. The tangent to the other branch lies in π. To sections by planes through k, the line joining the nodes, correspond the cubics of the pencil $8A$.

Twelve of these have a node, and the corresponding sections are tangent to the surface. In particular, to the sections by the planes kd_1, kd_3, ke_1, ke_2, ke_3 correspond the cubics of the pencil determined by B_1, B_3, E_1, E_2, E_3. To A_0, the ninth base point of the pencil of cubics, corresponds the direction of k at O.

In what follows it will be convenient to refer to the image of the triple point Q, that is, the sextic $8A^2B_2B_2'B_1B_3E_1E_2E_3$, as s; and to the image of the unode O, that is, the cubic $8AB_2B_2'$, as ϕ. The number of base curves of a general pencil that have an additional node, that is, the class of the surface, is 38. For we have in the first place $3(12-1)^2$ less 39 for each quadruple point, and less 7 for the double point B_2; that is, 44. But a general pencil of planes contains one section through each node. To the section through O corresponds in the pencil of plane base curves a nonic of the net above described and ϕ. This nonic meets ϕ at B_2 and in two more points which are to be deducted. To the section through Q corresponds a sextic of the net $8A^2B_2B_2'$. The two additional points in which it meets s are to be deducted.

But a further reduction of 2 is necessary on account of this degenerate duodecimic. The curves of the pencil have a common tangent at B_2'. When a pencil of curves have a common tangent at a base point the curve of the pencil that has a double point there counts for 2, or reduces by 2 the number of curves of the pencil that have an additional node elsewhere. In the present case this is the degenerate curve composed of s and the variable sextic. Hence the class of the quintic is 38. The Jacobian of a general net of base curves contains the factors s and ϕ. For the net contains two pencils of degenerate duodecimics, one composed of s and a variable sextic, the other of ϕ and a variable nonic. If we remove these factors, it appears that the image of the curve of contact of a general tangent cone is of order 24 and has 8-fold points at $A_1, \cdots, A_8$, a triple point at B_2, and simple points at B_2', B_1, B_3, E_1, E_2, E_3. Of the intersections of two such curves 527 are obvious at the base points. The class of the surface accounts for 38, leaving 11 to be found. We must, in fact, account for 7 more intersections of such a curve with s and 4 more with ϕ. Now it can be shown independently that it (the curve of order 24) is tangent to s at B_2' and meets s in 6 other fixed points, and ϕ in 4 other fixed points, apart from the base points. This accounts for the remaining 11 intersections of two such curves, and verifies the class of the surface. A brief description of the method of establishing this result is desirable.

Through any point P of space pass a pencil of nodal sections whose axis is PQ, a similar pencil whose axis is PO, and a unique section by the plane POQ. We have seen that the image of the last section is a cubic ψ of the pencil $8A$. To a plane section through PQ corresponds a sextic s' of the net $8A^2B_2B_2'$. To a plane section through PO corresponds a nonic n of the net $8A^3B_1B_2B_3E_1E_2E_3$. Since any three linearly independent curves of a net determine it and its jacobian, we may use to determine a general net of base curves ss', $s\phi\psi$ and $n\phi$. For a given net ψ is unique; but s' (and similarly n) is any one of a pencil. Moreover s' can be written $\alpha s+\beta\phi\psi+\gamma\phi^2$. We then find that $zJ=12s\phi zj$, where J is the jacobian, and j is of order 24, and is the image of the curve of contact of the corresponding tangent cone.

$$\begin{aligned} zj = {} & n[s'\psi s_x\phi_y + s'\phi s_x\psi_y + \phi\psi s_y s_x' + s\phi s_x'\psi_y - s'\psi s_y\phi_x - s'\phi s_y\phi_x \\ & - \phi\psi s_x s_y' - s\phi s_y'\psi_x - ss'\phi_x\psi_y + ss'\phi_y\psi_x] - ss'[\psi n_x\phi_y \\ & + \phi n_x\psi_y - \psi n_y\phi_x - \phi n_y\psi_x] + s\phi\psi[n_x s_y' - n_y s_x']. \end{aligned}$$

Putting $\phi=\psi=0$ we see that this expression does not vanish; and therefore the curve of order 24 does not pass through A_0, the ninth base point of the pencil of cubics $8A$. We wish to show that j is tangent to s at B_2'. Since j vanishes at B_2', the values of $(zj)_x$, $(zj)_y$, and $(zj)_z$ at B_2' are the coordinates of the tangent to j at that point. Since ϕ, s, and j vanish at B_2', the only terms in the three partial derivatives of zj that do not vanish at B_2' will arise from the terms $n\psi[s's_x\phi_y+\phi s_y s_x' - s's_y\phi_x - \phi s_x s_y']$. Omitting the value of $n\psi$ at B_2', which is a common factor in the derivatives, we have, at B_2',

$$\begin{aligned} (zj)_x : (zj)_y : (zj)_z = {} & s_x(s_x'\phi_y - s_y'\phi_x) \\ & : s_y(s_x'\phi_y - s_y'\phi_x) : [s_z'(s_x\phi_y - s_y\phi_x) - \phi_z(s_x s_y' - s_y s_x')]. \end{aligned}$$

Since $s'=\alpha s+\beta\phi\psi+\gamma\phi^2$, and since ϕ vanishes at B_2', the last expression at B_2' reduces to $\alpha s_z(s_x\phi_y - s_y\phi_x)$. And $s_x'\phi_y - s_y'\phi_x$ at that point reduces to $\alpha(s_x\phi_y - s_y\phi_x)$. Hence j is tangent to s at B_2'. We wish to show further that j meets s in 6 fixed points and n in 4 fixed points apart from the base points. In fact j meets s in the 6 points that are invariant under I_{17}. For n meets s only at the base points. Hence where zj (and therefore j) meets s apart from the base points we have

$$s'\psi s_x\phi_y + s'\phi s_x\psi_y + \phi\psi s_y s_x' - s'\psi s_y\phi_x - s'\phi s_y\psi_x - \phi\psi s_x s_y' = 0.$$

If we consider the jacobian of s', $\phi\psi$, and s, we find as above that it meets s, apart from the base points, where the last equation is satisfied, that is, where j meets s. But s' is a linear function of s, $\phi\psi$, and ϕ^2. So the jacobian of s', $\phi\psi$, and s is simply the locus of invariant points of I_{17} with the factor ϕ^2 added. This locus is of order 9, has triple points at $A_1, \cdots, A_8$ and meets s in 6 other points. This amounts to saying that in any pencil of sextics of the net $8A^2B_2B_2'$ there are 6 curves each tangent to s at one of 6 fixed points. Similarly s' meets ϕ only at the base points, and hence where j meets ϕ aside from those points we have

$$n(\psi s_x\phi_y - \psi s_y\phi_x - s\phi_x\psi_y + s\phi_y\psi_x) - s\psi(n_x\phi_y - n_y\phi_x) = 0.$$

If we consider the jacobian of n, $s\psi$, and ϕ, we find that it meets ϕ aside from the base points, where this equation is satisfied, that is, where j meets ϕ. Now n and $s\psi$ are two linearly independent nonics of the net $8A^3B_1B_2B_3E_1E_2E_3$. But in any pencil of nonics belonging to the web $8A^3B_1B_2B_3$ are 4 curves each tangent to ϕ at one of 4 fixed points, independent of the pencil considered. For the nonics of the web are the images of the plane sections of the rational sextic surface with which we started, and the cubic ϕ is the image of a double line, d_2, of the surface on which there are 4 pinch points. Therefore the jacobian of n, $s\psi$, and ϕ (and hence j) meets ϕ in these 4 points. For the jacobian of u, v, w meets $w=0$ once at each point of contact of a curve of the pencil $u+kv=0$ with $w=0$.*

The facts just established have an interesting geometrical significance. We have seen that the cones whose vertex is Q and which have the generators k, d_1, and d_3 meet the sextic surface in residual skew quintic curves, whose images are the sextics of the net $8A^2B_2B_2'$. By the quadratic transformation these cones become planes through Q meeting the quintic surface in sections whose images are the same net of sextics. In particular the pencil of cones which are tangent along k to a given plane ρ of the pencil k become a pencil of planes whose axis is the intersection

* Hilton, *Higher Plane Curves*, p. 110.

of ρ and σ, the plane $d_1d_2d_3$. A general plane ρ of k meets the sextic surface in a residual cubic whose image is a cubic of the pencil $8AA_0$. The plane ρ is tangent to the sextic at two generally distinct points on k corresponding to the points M and N, conjugate in I_{17}, in which the latter cubic meets the sextic s, the image of the triple line of the sextic surface and of the triple point Q of the quintic surface. Therefore the cones of the pencil tangent to ρ along k meet the sextic surface in space curves which all pass through these 2 points on k corresponding to M and N, and each of which passes simply through Q in a variable direction tangent to the plane kd_2 (called π), which is of course the direction of the second generator of the cone that lies in π.

Likewise, the quadratic transformation gives a pencil of plane sections of the quintic surface, each having a triple point at Q composed of a tacnode corresponding to M and N, the common tangent being the intersection of ρ and σ, and a variable third linear branch tangent to π; the latter being the transform of the element of direction just mentioned, and corresponding to the passage of the image curve (sextic) through B_2'. For 6 positions of ρ, the points of tangency of ρ on k, and the points M and N in the plane representation coincide. All the plane sextics of the pencil are tangent to s at the consecutive points MN. The resulting plane sections of the pencil of the quintic surface have each a triple point at Q composed of a cusp of the second kind and a variable linear branch tangent to π. The common cuspidal tangent is of course the intersection of ρ and σ. One plane of each of these 6 pencils passes through any point P of space.

This accounts for the fact noted above that the jacobian of a general net of base curves passes through the corresponding 6 points on s. One sextic of the plane pencil has a double point at the consecutive points MN. The corresponding plane section of the quintic has an oscnode and a third linear branch at Q. The pencil of cones tangent along k to the plane kd_2 (π) and containing d_1 and d_3 is interesting. The plane π is tangent to the sextic surface at L where the residual line in π meets k. Hence these cones meet the sextic surface in curves which are tangent to π at L and osculate π at Q, the tangent at Q being k. The images of these curves are the sextics of the pencil of the net $8A^2B_2B_2'$ tangent to s at B_2 and B_2'. To the passage of the image curve through B_2' tangent to s corresponds the passage of the space curve through Q tangent to k. By the quadratic transformation such an element at Q becomes a cusp of the first kind, the cuspidal tangent being the line $\pi\sigma$. Therefore the resulting curves of plane sections of a pencil of the quintic surface have each an ordinary cusp at Q and a linear branch, both tangent to the line $\pi\sigma$.

We have seen that the jacobian of a general net of base curves passes through 6 fixed points on s and is tangent to s at B_2'. Hence the curve of contact of the corresponding tangent cone (of order 20) has at Q 6 linear branches tangent to fixed directions in σ and a cusp whose tangent is the line $\pi\sigma$. The other 5 intersections of this curve with σ take place one on each of the 5 lines of the surface lying in that plane. Hence a general tangent cone has at Q a multiple edge equivalent to 27 double edges and one cuspidal edge. Similarly, since a plane through O is invariant as a whole under the quadratic transformation, the sections of the sextic surface by such planes, whose images are the nonics of the net $8A^3B_1B_2B_3E_1E_2E_3$, become the sections of the quintic surface by the same planes. A general plane through O meets the sextic surface in a curve with a double point on d_2, and the resulting section of the quintic by the same plane has a tacnode at O. The sextic surface has 4 pinch points on d_2 whose image is the cubic $8AB_2B_2'$, called ϕ above. Hence on this cubic are 4 points where the pencil of nonics of the above mentioned net that pass through the point are tangent there to the cubic.

The corresponding sections of the sextic surface have ordinary cusps on d_2, and the resulting sections of the quintic have cusps of the second kind at O instead of tacnodes. One nonic of the plane pencil through such a point has a double point there. The corresponding section of the sextic has a tacnode on d_2, and the resulting section of the quintic has an oscnode at O. We have seen that the jacobian of a general net of base curves of the quintic surface passes through these 4 points on ϕ. Hence the curve of contact of the corresponding tangent cone passes 4 times through O tangent to fixed directions in the plane kd_2 (π). 9 intersections of this curve with π occur at Q, 3 of them accounted for by the cusp at Q, and the remaining 3 intersections occur on the conic whose image is B_2. Thus a general tangent cone has at O an ordinary quadruple edge equivalent to 6 double edges. This accounts satisfactorily for the Plücker characteristics of the cone. For its genus, which is the genus of j, the image of its curve of contact, is 26, its order is 20, and its class 38. Hence it has 52 cuspidal edges and the equivalent of 93 double edges. One of the cuspidal edges is at Q. In case of a general quintic surface the tangent cone would have 60 double edges. We have seen that Q accounts for 27 and O for 6, which make up the 33 additional required by the surface in question.

From Q a cone of lines can be drawn tangent elsewhere to the surface. The image of the curve of contact is the locus of invariant points in I_{17}. Similarly the image of the curve of contact of tangents drawn to the surface from O is of order 18, for it loses the factor s. The image of the parabolic curve, that is, the locus of cusps of the base curves, is obtained by deducting one of these base curves from the jacobian of a general net of curves j. Since the intersections of a j with s and ϕ are all fixed, they are factors of the jacobian in question. Hence the image of the parabolic curve is of order 48, and has 16-fold points at $A_1, \cdots, A_8$, and a 4-fold point at B_2, but does not pass through B_2', or any of the other base points. This shows that in a pencil of plane sections through any of the 6 lines of the surface the residual quartic is never tangent to the axis apart from the nodes. The sextics which have double points at $A_1, \cdots, A_8$ but do not pass through B_2 give on the surface octavic curves of genus 2. Such a sextic may be composed, in 28 ways, of the line joining 2 of the points A and the rational quintic having double points at the other six and passing through the first two. The line and the quintic meet 3 times on the locus of invariant points in I_{17}. The 2 additional points in which the line meets s, the image of Q, correspond in I_{17} to the 2 similar points in which s is met by the quintic. Since two points on s that are conjugate in I_{17} give the same direction at Q in σ, we have on the surface 28 pairs of rational quartics, each quartic passing twice through Q and once through O, and the two quartics of a pair being tangent to each other twice at Q. Similarly the sextic may be composed of the conic through five of the points A and the quartic having double points at the other three and passing through the first five. So we get 56 more pairs of such quartics. Finally the rational quartics corresponding to $A_1, \cdots, A_8$ are of this type. With the quartic corresponding to A_i is paired the rational quartic whose image is the sextic that has a triple point at A_i and double points at the other 7 points A. This sextic corresponds to A_i in I_{17}.

Mention may be made of a few slightly specialized forms of the surface. In setting up the plane system we take 8 general points A which determine A_0, and a point B_2 which determines B_2' and the cubic $8AB_2B_2'$, called ϕ. Any proper sextic of the net $8A^2B_2B_2'$ can be taken for s. A proper duodecimic $8A^4B_2^2$ tangent to ϕ at B_2' meets s in $B_1B_3E_1E_2E_3$. Three of the latter points determine the remaining two. If the duodecimic is tangent to s, two of the five corresponding lines in σ coincide and the surface acquires a conic node on the coincident lines. If $E_1E_2E_3$ are taken consecutive on s, three lines in σ become coincident, and the new node is a binode reducing the class by 3. If B_2 is taken on the line joining two of the points $A_1, \cdots, A_8$, this line with s and the quintic which has double points at the remaining six points A and passes through the first two and necessarily through B_2' constitutes the image of a plane section. To the quintic corresponds on the surface a rational cubic having a node at Q. To the line corresponds a conic tangent to one of the nodal tangents of the cubic at Q. If the line determined by two points A contains B_2 and one of the 5 remaining simple points this conic becomes the corresponding line in σ and one other line. If three of the points $A_1, \cdots, A_8$ lie on a line, the surface has a new conical node. To the conic through the other five points A corresponds a rational plane quartic tangent to k at O, and having a tacnode at Q and a node at the new node of the surface. The images of the plane sections through Q and the new node are the quintics forming a pencil which have double points at the five non-collinear points A and simple points at the other three, and which pass through B_2 and hence B_2'. The images of the plane sections through O and the new node are the curves of a pencil of order 8 having triple points at the five non-collinear points A, double points at the other three, and passing through B_2 and the five simple base points. A similar situation arises if six of the points $A_1, \cdots, A_8$ lie on a conic.

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