ON COMPLEX METHODS OF SUMMABILITY*

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1. *Introduction.* When a method of summability‡ evaluates a complex sequence $\{s_n\}$ to L, it is of interest to know if that method evaluates $\{\mathcal{R}(s_n)\}$ to $\mathcal{R}(L)$, if it evaluates $\{\mathcal{J}(s_n)\}$ to $\mathcal{J}(L)$, and if it evaluates $\{\bar{s}_n\}$ to $\bar{L}$.§ For linear methods of summability,|| these three questions are easily shown to be equivalent.

To simplify our discussion, we introduce the two following definitions. *A method of summability has property* A *if, corresponding to each sequence* $\{s_n\}$ *which it evaluates, the sequence* $\{\mathcal{R}(s_n)\}$ *is evaluated to the real part of the value of* $\{s_n\}$. *A method of summability has property* B *if, corresponding to each bounded sequence* $\{s_n\}$ *which it evaluates, the sequence* $\{\mathcal{R}(s_n)\}$ *is evaluated to the real part of the value of* $\{s_n\}$.

2. *Failure of Property* B. That a linear regular method may fail to have property B, and hence a fortiori fail to have property A, follows easily from a consideration of the transformation¶

$$(1) \qquad \sigma_n = \tfrac{1}{2}[1 - (-1)^n i]\, s_{n-1} + \tfrac{1}{2}[1 + (-1)^n i]\, s_n$$

which assigns to a given sequence $\{s_n\}$ the value $\lim \sigma_n$ when this limit exists. The bounded sequence $\{x_n\}$ defined by $x_n = 1 + (-1)^n i$ is evaluated to 0 by (1); but $\{\mathcal{R}(x_n)\}$ is evaluated to 1, $\{\mathcal{J}(x_n)\}$ is evaluated to -1, and $\{\bar{x}_n\}$ is evaluated to 2. This

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‡ By a *method of summability*, we mean simply a rule which assigns to each given sequence (or series) of complex numbers either no value or a single value. For example, if we agree to assign to the complex sequence $\{s_n\}$, where $s_n = u_n + iv_n$, u_n and v_n real, the value $3+4i$ if $v_n \neq 0$ for some n and the value 3 if $v_n = 0$ for all n, we have a method of summability.

§ If $w = u + iv$, where u and v are real, we use $\mathcal{R}(w)$, $\mathcal{J}(w)$, and $\bar{w}$ to denote respectively u, iv, and the conjugate $u - iv$ of w.

|| For definitions of *linearity*, *regularity*, etc., and for necessary and sufficient conditions for regularity, see an expository paper by W. A. Hurwitz, this Bulletin, vol. 28 (1922), pp. 17–36, and the references there given.

¶ Corresponding to a given sequence $s_1, s_2, s_3, \cdots$, we define $s_0 = 0$.

example is of especial interest in that $\{\mathfrak{R}(x_n)\}$ not only fails to be summable to the real part of the value of $\{x_n\}$, but is actually summable to a different value.

It will appear later (§8) that a linear regular method may have property B and fail to have property A.

3. *Necessary Condition for Properties* A *and* B. The following two theorems, of which the proofs are immediate, give conditions which are necessary in order that a method may have property A or property B.

THEOREM 1. *If a method having property* A *evaluates a real sequence, the value assigned must be real.*

THEOREM 2. *If a method having property* B *evaluates a real bounded sequence, the value assigned must be real.*

The converses of these theorems do not hold as we shall now show by giving an example of a linear regular transformation which assigns a real value to each real sequence which it evaluates, but which nevertheless fails to have properties A and B. The transformation is

$$\sigma_n = \tfrac{1}{2}(1 - i)s_{n-1} + \tfrac{1}{2}(1 + i)s_n. \tag{2}$$

To show that (2) assigns a real value to each real sequence which it evaluates, let $\{x_n\}$ be a real sequence evaluated by (2) to L_1+iL_2. Then $\lim \frac{1}{2}(x_n+x_{n-1})=L_1$ and $\lim \frac{1}{2}(x_n-x_{n-1})=L_2$, and on adding and subtracting we find $x_n\rightarrow L_1+L_2$ and $x_{n-1}\rightarrow L_1-L_2$; hence $L_1+L_2=L_1-L_2$ so that $L_2=0$ and the reality of the value is established. To show that (2) does not have properties A and B, we consider the sequence $\{y_n\}$ defined by

$$y_{4p-3} = 1 - i,\ y_{4p-2} = 1 + i,\ y_{4p-1} = -1 + i,\ y_{4p} = -1 - i, \qquad (p = 1, 2, 3, \cdots).$$

It is easily verified that (2) evaluates $\{y_n\}$ to 0, and that (2) fails to evaluate $\{\mathfrak{R}(y_n)\}$; hence, since $\{y_n\}$ is bounded, (2) fails to have properties A and B.

4. *Complex Transformations and Associated Real Transformations.* We now consider methods of summability which involve transformations (G) defined as follows. Let T be a metric set having a limit point t_0 not belonging to T, and let complex

functions $a_k(t)$, $k=1, 2, 3, \cdots$, be defined for all t in T. If a sequence s_n is such that

$$\text{(G)} \qquad \sigma(t) = \sum_{k=1}^{\infty} a_k(t) s_k$$

converges for all t in T, and if

$$\lim_{t \to t_0(T)} \sigma(t) = L,$$

then $\{s_n\}$ is said to be summable by the method or transformation (G), or simply summable (G), to L.

Corresponding to a given (G), we define $b_k(t)$ and $c_k(t)$ to be the real functions determined by the equations

$$a_k(t) = b_k(t) + i c_k(t).$$

The transformation

$$\mathfrak{R}\text{(G)} \qquad \sigma(t) = \sum_{k=1}^{\infty} b_k(t) s_k$$

may be called the *associated real transformation* of (G). Using the well known necessary and sufficient conditions for regularity of (G), we obtain the result: *If* (G) *is regular, then its associated real transformation* $\mathfrak{R}$(G) *is also regular.*

5. *Properties of Transformations.* The following two theorems serve to establish the equivalence of certain problems involving properties of (G).

THEOREM 3. *In order that* (G) *may have property* A, *it is necessary and sufficient that* $\mathfrak{R}$(G) *include* (G).

THEOREM 4. *In order that* (G) *may have property* B, *it is necessary and sufficient that* $\mathfrak{R}$(G) *include* (G) *over the set of all bounded sequences.**

We will now prove Theorem 3; the same method, together with the assumption that all sequences considered are bounded, furnishes a proof of Theorem 4.

To establish sufficiency, let $\mathfrak{R}$(G) include (G) and let (G) evaluate a given sequence $\{s_n\}$ to L; we are to show that (G)

* We say that a method includes a second method over a set S of sequences if each element of S which is summable by the second method is also summable to the same value by the first method.

evaluates $\{\mathfrak{R}(s_n)\}$ to $\mathfrak{R}(L)$. It then follows from the definition of summability (G) that the series

$$\sum_{k=1}^{\infty} a_k(t)s_k, \quad \sum_{k=1}^{\infty} b_k(t)s_k$$

both converge over T, and that the value of each approaches L as $t \to t_0$; hence $\sum_{k=1}^{\infty} c_k(t)s_k$ converges over T, and its value $\to 0$ as $t \to t_0$. Since $b_k(t)$ and $c_k(t)$ are real, it follows that the series

$$\sum_{k=1}^{\infty} b_k(t)\mathfrak{R}(s_k), \quad \sum_{k=1}^{\infty} c_k(t)\mathfrak{R}(s_k)$$

both converge over T, and that their values approach $\mathfrak{R}(L)$ and 0 respectively as $t \to t_0$. Hence the series

$$\sum_{k=1}^{\infty} a_k(t)\mathfrak{R}(s_k) \equiv \sum_{k=1}^{\infty} [b_k(t) + ic_k(t)]\mathfrak{R}(s_k)$$

converges over T and its value approaches $\mathfrak{R}(L)$ as $t \to t_0$. Thus sufficiency is proved.

To establish necessity, let (G) have property A, and let $\{s_n\}$ be a sequence which (G) evaluates to L; we are to show that $\mathfrak{R}$(G) evaluates $\{s_n\}$ to L. We find that the series

$$\sum_{k=1}^{\infty} a_k(t)s_k, \quad \sum_{k=1}^{\infty} a_k(t)\mathfrak{R}(s_k)$$

both converge over T, and their values approach L and $\mathfrak{R}(L)$ respectively as $t \to t_0$. Hence $\sum_{k=1}^{\infty} a_k(t)\mathfrak{J}(s_k)$ converges over T and approaches $\mathfrak{J}(L)$ as $t \to t_0$. Since $\mathfrak{R}(s_k)$ is real and $\mathfrak{J}(s_k)$ is pure imaginary, it follows that the series

$$\sum_{k=1}^{\infty} b_k(t)\mathfrak{R}(s_k), \quad \sum_{k=1}^{\infty} b_k(t)\mathfrak{J}(s_k)$$

both converge over T and that their values approach $\mathfrak{R}(L)$ and $\mathfrak{J}(L)$ respectively as $t \to t_0$. Hence the series

$$\sum_{k=1}^{\infty} b_k(t)s_k \equiv \sum_{k=1}^{\infty} b_k(t)[\mathfrak{R}(s_k) + \mathfrak{J}(s_k)]$$

converges over T and approaches $\mathfrak{R}(L) + \mathfrak{J}(L) = L$ as $t \to t_0$. Thus necessity is proved.

6. THEOREM 5. *If* (G) *satisfies the conditions*

$$\sum_{k=1}^{\infty} |c_k(t)| \text{ converges over } T, \tag{3}$$

$$\lim_{t \to t_0(T)} \sum_{k=1}^{\infty} | c_k(t) | = 0, \tag{4}$$

then (G) *and* $\mathcal{R}$(G) *are equivalent over the set of all bounded sequences.**

Let $\{s_n\}$ be any bounded sequence. It follows from (3) and (4) that the series $\sum_{k=1}^{\infty} c_k(t)s_k$ converges over T and that its value approaches 0 as $t \to t_0$. Hence if either of the series

$$\sum_{k=1}^{\infty} a_k(t)s_k, \qquad \sum_{k=1}^{\infty} b_k(t)s_k$$

converges over T, the other must also converge over T and we may write

$$\sum_{k=1}^{\infty} a_k(t)s_k = \sum_{k=1}^{\infty} b_k(t)s_k + i \sum_{k=1}^{\infty} c_k(t)s_k.$$

It follows that if the value of either of the series approaches a limit as $t \to t_0$, the value of the other series must approach the same limit and the theorem is proved.

7. *Sufficient Condition for Property* B. Combining Theorems 4 and 5, we obtain the following result.

THEOREM 6. *If* (G) *satisfies* (3) *and* (4), *then* (G) *has property* B.

Since (G) is linear, the preceding theorem may be amplified to produce the following theorem.

THEOREM 7. *If* (G) *satisfies* (3) *and* (4), *and* $\{s_n\}$ *is a bounded sequence which* (G) *evaluates, say to* L, *then* $\{\mathcal{R}(s_n)\}$, $\{\mathcal{J}(s_n)\}$ *and* $\{\bar{s}_n\}$ *are evaluated by* (G) *to* $\mathcal{R}(L)$, $\mathcal{J}(L)$, *and* $\bar{L}$ *respectively.*

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* The conditions (3) and (4) are of course not sufficient to ensure regularity of (G); hence this theorem and its applications give information concerning non-regular transformations. On the other hand, (3) is satisfied by every regular (G), and (4) is satisfied by every regular (G) which satisfies the important and useful condition $\lim_{t \to t_0(T)} \sum_{k=1}^{\infty} |a_k(t)| = 1$.

8. *Examples.* We will now give an example of a linear regular transformation of the form (G) which has property B and fails to have property A. This example shows that Theorems 2, 4, 5, and 7 do not hold if the word "bounded" is omitted, and that B cannot be replaced by A in Theorem 6. The transformation is

$$\text{(H)} \quad \sigma_n = \frac{1}{2}\left[1 - \frac{(-1)^n i}{\log(n+1)}\right]s_{n-1} + \frac{1}{2}\left[1 + \frac{(-1)^n i}{\log(n+2)}\right]s_n,$$

which assigns to a given sequence $\{s_n\}$ the value $\lim \sigma_n$ when this limit exists. That (H) is of the form (G) is seen by taking (T) to be the set of positive integers, t_0 to be the symbolic limit point $+\infty$, and by writing σ_n for $\sigma(n)$. Evidently (H) satisfies (3) and (4), (H) has property B, and (H) and $\mathcal{R}$(H) are equivalent over the set of all bounded sequences. However, we find that (H) evaluates the unbounded sequence

$$x_n = 1 + (-1)^n i \log(n+2)$$

to 0; and further that (H) evaluates $\{\mathcal{R}(x_n)\}$ to 1, $\{\mathcal{J}(x_n)\}$ to -1, $\{\bar{x}_n\}$ to 2, and the real sequence $\{i\mathcal{J}(x_n)\}$ to the imaginary value $-i$. Finally, $\mathcal{R}$(H) evaluates $\{x_n\}$ to 1.

This example shows that (4) is not sufficient to ensure mutual consistency of a regular (G) and its associated real transformation $\mathcal{R}$(G).

The condition (4) is not necessary in order that (G) and $\mathcal{R}$(G) may be equivalent, or that (G) may have properties A and B. In fact, it is easy to show that the regular transformation

$$\text{(J)} \qquad \sigma_n = [i + (-1)^n i]s_{n-1} + [1 - i - (-1)^n i]s_n,$$

which fails to satisfy (4), is equivalent to its associated real transformation $\mathcal{R}$(J) which obviously has properties A and B.

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