FACTORABILITY OF NUMERICAL FUNCTIONS*

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1. *Factorability.* To make this note self-contained we recall a few definitions and theorems from previous papers 1, 2. The new results complete a previous theory in an interesting detail. References are in §4. As some of the conclusions are rather unexpected, we shall state one in full in ordinary notation while giving the definitions; it is restated in §2 and proved in §3 by the symbolic method appropriate to the subject.

If $f(x)$ is single-valued and finite for integer values >0 of x, $f(x)$ is called a numerical function of x. If $k(x)$ is a numerical function of x having the property that

$$k(mn) = k(m)\ k(n)$$

whenever m, n are coprime integers, <0, $k(x)$ is said to be factorable. If $g(x)$ is a numerical function, $g(x)$ is said to be regular or irregular according as $g(1) \neq 0$ or $g(1) = 0$.

Let $f(x)$, $g(x)$ be regular numerical functions. Define $h(n)$ by

$$h(n) = \sum f(d)g(t) \tag{1}$$

for all integers n, where the sum refers to all pairs (d, t) of positive divisors of n such that $dt = n$. Then, by the definition, $h(x)$ is a numerical function of x. Among other theorems we shall prove the following.

If $h(x)$ is factorable, then either both of $f(x)$, $g(x)$ are factorable or neither of $f(x)$, $g(x)$ is factorable.

The converse is false, but *if both of $f(x)$, $g(x)$ are factorable, $h(x)$ is factorable.*

As in the papers 1, $\cdots$, 4 cited below, we shall write (1) symbolically, $h = fg$, and refer to f instead of $f(x)$ as a numerical function.

The unit numerical function is the unique η such that $\eta f = f$ for all numerical functions f, and η is defined by $\eta(1) = 1$, $\eta(n) = 0$ $(n > 1)$. If and only if f is regular, there exists[3] a unique

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regular numerical function, denoted by f^{-1}, such that $ff^{-1}=\eta$. It is necessary to recall that the symbolic multiplication in $fg \cdots h$, where $f, g, \cdots, h$ are numerical functions, is associative and commutative;[2,4] it is also distributive over the relevant addition,[6] but this is not required here. Further, if r is an integer >0, f^r denotes the product (in the above sense) of r factors f; if s is a rational number, $f^s=g$ is interpreted as $f^a=g^b$, where a, b are coprime integers (positive or negative) and $s=a/b$. If in the last one of a, b is negative, say b, we may write $f^a=g^b$ in the equivalent forms $f^a g^{-b}=\eta$, b positive.

If and only if f is regular and factorable it has a generator[2,5] of the form

$$\Gamma(f) \equiv 1 + F_1(p)z + F_2(p)z^2 + \cdots,$$

where the series need not terminate, and the F's are single-valued functions of p. For details we must refer to previous papers.[5] If $a, b, \cdots$, are rational numbers, and

$$\Gamma(f), \Gamma(g), \cdots, \Gamma(h)$$

are the generators of the regular factorable numerical functions $f, g, \cdots, h$, the generator[5] of $f g \cdots h$ is that branch of the formal expansion of

$$(\Gamma(f))^a(\Gamma(g))^b \cdots (\Gamma(h))^c$$

whose term free of p, z is 1. If $f, g, \cdots, h$ are regular, so also is $fg \cdots h$.

2. *Theorems*. The first is implicit in the previous papers, but we state it here for completeness, as it is used in the proofs.

THEOREM 1. *If f is a regular numerical function and r is a rational number, f^r is factorable if and only if f is factorable. If $f, g, \cdots, h$ are regular factorable numerical functions and $a, b, \cdots, c$ are rational numbers, $f^a g^b \cdots h^c$ is factorable.*

LEMMA. *If f, g, h are numerical functions such that f, g are regular and $fg=h$, then either both of f, g are factorable or neither of f, g is factorable.*

This is used in proving the next, which includes the lemma.

THEOREM 2. *If $f, g, \cdots, h, k$ are numerical functions, of which $f, g, \cdots, h$ are regular and k is factorable, such that*

$$f^a g^b \cdots h^c = k,$$

where $a, b, \cdots, c$ *are rational numbers, then all or none of* $f, g, \cdots, h$ *are factorable.*

It remains to be seen that the last is a genuine theorem and not a true but vacuous statement.

THEOREM 3. *There exist an infinity of* $f, g, \cdots, h$ *satisfying either one of the conclusions of Theorem* 2.

3. *Proofs.* Theorem 1 is a mere restatement of the last paragraph of §1. Theorem 2 then follows by applying Theorem 1 and the Lemma to $f^a g^b \cdots h^c$ considered as $f^a(g^b \cdots h^c)$, repeating the argument for $g^b \cdots h^c$, and so on. In the Lemma we have $g = f^{-1}h$, h factorable. Hence by Theorem 1, if f is factorable, so also is $f^{-1}h$, and therefore g is factorable. If f is not factorable and g is factorable, from $f = g^{-1}h$ we get the contradiction that f is factorable. In Theorem 3 the part concerning the first conclusion of Theorem 2 follows from Theorem 1. The second part follows from Theorem 2 and the first part of Theorem 1, applied to $f^c h^a$, $f^{-c}h^b$, where h is factorable and f is not factorable, and a, b, c are rational numbers. For, the product of $f^c h^a$ and $f^{-c}h^b$, neither of which is factorable, is h^{a+b}, which is factorable.

4. *References.*

(1) *Outline of a theory of arithmetical functions in their algebraic aspects*, Journal of the Indian Mathematical Society, vol. 17 (1928). This paper contains references to my earlier papers on the same topic.

(2) *An arithmetical theory of certain numerical functions*, University of Washington Publications in Science, vol. 1, No. 1, 1915.

(3) *A certain inversion in the theory of numbers*, Tôhoku Mathematical Journal, vol. 17 (1920), pp. 221–231.

(4) *Extensions of Dirichlet multiplication and Dedekind inversion*, this Bulletin, vol. 28 (1922), pp. 111–122.

(5) *Euler algebra*, Transactions of this Society, vol. 25 (1923), pp. 135–154.

(6) *Algebraic Arithmetic.* Colloquium Publications of this Society, vol. 7, 1927.

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