ON THE TRIGONOMETRIC EXPANSION OF ELLIPTIC FUNCTIONS*

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1. *Introduction.* The problem of expressing an elliptic function in terms of infinite sums of trigonometric functions has been treated by Hermite,† Briot and Bouquet,‡ A. C. Dixon§ and others. In the present paper we treat the same problem from the point of view of Cauchy's residue theorem in function theory, which is also Briot and Bouquet's starting point, but we differ from these authors in that the integrand we use leads to an expansion for an elliptic function which is valid in an arbitrarily wide, but finite, strip of the complex plane, and which contains certain classical results as special cases. An interesting feature of our expansion is that it yields quite directly the Fourier series development of the function. Some examples of this property are indicated as an illustration of the applicability of our formula. It should be noted that the integrand used was first given by F. Gomes Teixeira¶ in another connection.

2. *Expansion of $f(z)$.* Let $f(z)$ be an elliptic function with periods π and $\pi\tau$, where $\tau=\mu+i\nu$, μ, ν real and $\nu>0$. Suppose that in a fundamental period cell $f(z)$ has k poles, $z=a_r$, $(r=1, 2, \cdots, k)$. Further, let the order of these poles be m_r, so that in the neighborhood of $z=a_r$, $f(z)$ has the Laurent expansion

$$(1)\quad f(z)=\frac{A_{m_r}^{(r)}}{(z-a_r)^{m_r}}+\cdots+\frac{A_2^{(r)}}{(z-a_r)^2}+\frac{A_1^{(r)}}{z-a_r}+P(z-a_r),$$

where $P(z-a_r)$ is a power series in $z-a_r$.

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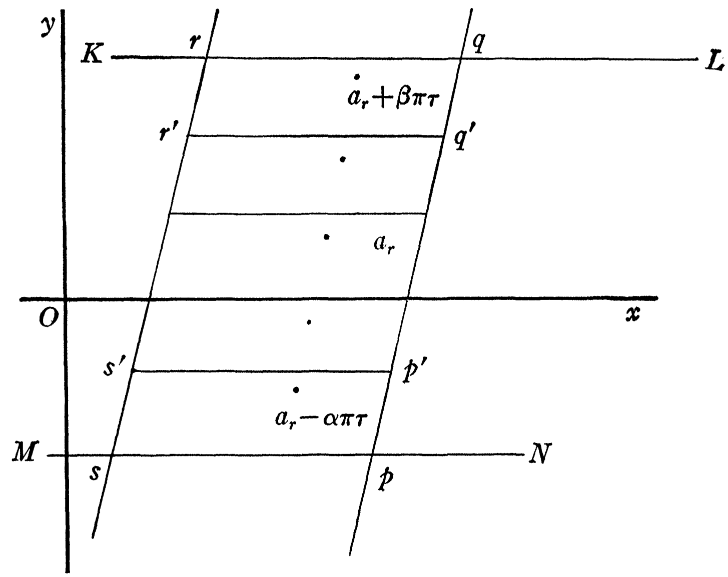
† C. Hermite, Annales de Toulouse, vol. 2 (1888). (See also Halphen, vol. 1, p. 461.)

‡ Briot and Bouquet, *Théorie des Fonctions Elliptiques*, 2d ed., 1875, p. 286.

§ A. C. Dixon, Quarterly Journal of Mathematics, vol. 34, p. 222.

¶ F. Gomes Teixeira, *Sur les séries ordonnées suivant les puissances d'une fonction donnée*, Journal für Mathematik, vol. 122, pp. 97–123.

Consider a parallelogram $pqrs$ consisting of $(\beta+1)$ period cells above the real axis and α below, so that we may write $|\overline{sp}| = \pi$ and $|\overline{pq}| = |\pi\tau|(\alpha+\beta+1)$. In this parallelogram, $f(z)$ has $(\alpha+\beta+1)k$ poles which may be represented by the affixes



$$z = a_r + m\pi\tau, \quad \begin{cases} r = 1, 2, 3, \cdots, k; \\ m = -\alpha, -\alpha+1, \cdots, -1, 0, 1, 2, \cdots, \beta. \end{cases}$$

The function of t

$$(2) \qquad \phi(t) \equiv \frac{f(t)e^{2it}}{e^{2it} - e^{2iz}} = f(t)\left\{\frac{1}{2} + \frac{1}{2i}\operatorname{ctn}(t-z)\right\}$$

has, in the parallelogram $pqrs$, poles at $t=z$ and $t=a_r+m\pi\tau$; the corresponding residues are easily calculated and found to be

$$(3) \quad \frac{1}{2i}f(z), \text{ and } \frac{1}{2}A_1^{(r)} + \frac{1}{2i}\sum_{s=1}^{m_r}\frac{A_s^{(r)}}{(s-1)!}\mathcal{D}_{\omega_{r,m}}^{(s-1)}\operatorname{ctn}(\omega_{r,m} - z),$$

where $\mathcal{D}^{(s-1)}$ is the differential operator of order $(s-1)$ and $\omega_{r,m}$ is the argument $a_r+m\pi\tau$ which is to be substituted after the differentiation has been performed.

On integrating $\phi(t)$ around the contour C of the parallelogram $pqrs$ we obtain, by Cauchy's theorem, the expression

$$
\begin{aligned}
f(z) &= \frac{1}{\pi}\int_{(C)} \frac{f(t)e^{2it}}{e^{2it}-e^{2iz}}\,dt \\
(4)\qquad &+ \sum_{s=1}^{m_r}\sum_{r=1}^{k}\sum_{m=-\alpha}^{\beta} \frac{A_s^{(r)}}{(s-1)!}\,\mathcal{D}_{\omega_{r,m}}^{(s-1)} \operatorname{ctn}(z-\omega_{r,m}).
\end{aligned}
$$

In writing (4) we have also used the fact that for an elliptic function the sum of its residues in any period cell vanishes.

The integral in (4) may be transformed into an infinite series in the following manner. Due to the fact that the integrand has the period π, the integrals along $\overline{pq}$ and $\overline{rs}$ cancel. Further, along $\overline{sp}$ we have

$$\frac{1}{e^{2it}-e^{2iz}} = \frac{1}{e^{2it}} + \frac{e^{2iz}}{e^{4it}} + \frac{e^{4iz}}{e^{6it}} + \cdots,$$

while along $\overline{rq}$,

$$\frac{1}{e^{2it}-e^{2iz}} = -\left(\frac{1}{e^{2iz}} + \frac{e^{2it}}{e^{4iz}} + \frac{e^{4it}}{e^{6it}} + \cdots\right).$$

It follows that our integral may be written in the form

$$\frac{1}{\pi}\left\{\int_{(\overline{sp})} f(t)\sum_{n=0}^{\infty} e^{2ni(z-t)}dt + \int_{(\overline{rq})} f(t)\sum_{n=1}^{\infty} e^{-2ni(z-t)}dt\right\}.$$

Interchanging the order of integration and summation, which is permissible, and using the notation

$$
(5)\qquad \begin{cases}
C_n = \dfrac{1}{\pi}\displaystyle\int_{(\overline{sp})} f(t)e^{-2nit}dt, & (n = 0, 1, 2, \cdots); \\[2ex]
C_{-n} = \dfrac{1}{\pi}\displaystyle\int_{(\overline{rq})} f(t)e^{2nit}dt, & (n = 1, 2, 3, \cdots),
\end{cases}
$$

we may write the formula (4) in the form

$$
\begin{aligned}
(6)\quad f(z) &= \sum_{n=-\infty}^{n=\infty} C_n e^{2niz} \\
&+ \sum_{s=1}^{m_r}\sum_{r=1}^{k}\sum_{m=-\alpha}^{m=\beta} \frac{A_s^{(r)}}{(s-1)!}\,\mathcal{D}_{\omega_{r,m}}^{(s-1)} \operatorname{ctn}(z-\omega_{r,m}).
\end{aligned}
$$

3. *Advantage of the Function $\phi(t)$.* The particular advantage in using the integrand $\phi(t)$ is that it leads in a natural manner to the expressions in (5) for the constants C_n and C_{-n}, all of which, with the exception of C_0, may be readily calculated. Thus, to compute C_n, integrate $f(t)e^{-2nit}$ around the contour $spp's'$ and apply Cauchy's theorem. Proceeding in the usual manner, it is readily found that the sum of the residues of the integrand, relative to the poles $t=a_r-\alpha\pi\tau$, has the value

$$q^{2n\alpha}\sum_{r=1}^{k}\sum_{s=1}^{m_r}\frac{(-2ni)^{s-1}A_s^{(r)}}{(s-1)!}e^{-2nia_r}.$$

Furthermore, the integrals along pp' and ss' cancel while

$$\int_{(\overline{p's'})} f(t)e^{-2nit}\,dt = -q^{-2r}\int_{(\overline{sp})} f(t)e^{-2nit}dt,$$

where, as usual, $q=e^{\pi i\tau}$.

We therefore find that

$$(7) \qquad C_n = \frac{q^{2n(\alpha+1)}}{1-q^{2n}}\sum_{r=1}^{k}\sum_{s=1}^{m_r}\frac{(-2i)^s n^{s-1}A_s^{(r)}}{(s-1)!}e^{-2nia_r}, \qquad (n=1,2,3,\cdots).$$

It should be noticed that C_0 is left undetermined.

In a similar manner, applying the residue theorem to the function $f(t)e^{2nit}$, using the parallelogram $r'q'qr$ as contour of integration, we find that

$$(8) \qquad C_{-n} = \frac{q^{2n(\beta+1)}}{1-q^{2n}}\sum_{r=1}^{k}\sum_{s=1}^{m_r}\frac{(2i)^s n^{s-1}A_s^{(r)}}{(s-1)!}e^{2nia_r}.$$

If now, (7) and (8) are substituted in (6) we obtain the result

$$(9) \qquad \left\{\begin{aligned} f(z) = C_0 &+ \sum_{r=1}^{k}\sum_{s=1}^{m_r}\sum_{n=1}^{\infty}\frac{(2i)^s n^{s-1}q^{2n}A_s^{(r)}}{(s-1)!(1-q^{2n})}\Big\{q^{2n\beta}e^{-2ni(z-a_r)} \\ &\qquad + (-1)^s q^{2n\alpha}e^{2ni(z-a_r)}\Big\} \\ &+ \sum_{r=1}^{k}\sum_{s=1}^{m_r}\sum_{m=-\alpha}^{m=\beta}\frac{A_s^{(r)}}{(s-1)!}\mathcal{D}_{\omega_{r,m}}^{(s-1)}\operatorname{ctn}(z-\omega_{r,m}),\end{aligned}\right.$$

which is valid in the strip bounded by the lines KL and MN. The width of this strip is, of course, determined by the values of the integers α and β.

4. *Some Special Cases.* The following special cases of (9) are of interest. First, suppose that all the poles of $f(z)$ are simple, so that $m_r=1$; further let $\alpha=0$, $\beta=0$. Then

$$
\begin{aligned}
f(z) &= C_0 + 4\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{q^{2n}A_1^{(r)}}{1-q^{2n}}\sin 2n(z-a_r) \\
&\quad + \sum_{r=1}^{k}A_1^{(r)}\operatorname{ctn}(z-a_r),
\end{aligned}
\tag{10}
$$

which is another form of the classical formula of decomposition

$$
f(x) = C_0 + \sum_{r=1}^{k}A_1^{(\ )}\frac{\vartheta_1'(z-a_r)}{\vartheta_1(z-a_r)},
\tag{11}
$$

where $\vartheta_1(z)$ is one of the Jacobi theta functions.

Again, let α and β become infinite; then since the absolute value of q is less than unity, it is easily seen that (9) becomes

$$
f(z) = C_0 + \sum_{r=1}^{k}\sum_{s=1}^{m_r}\sum_{m=-\infty}^{m=\infty}\frac{A_s^{(r)}}{(s-1)!}\mathcal{D}_{\omega_{r,m}}^{(s-1)}\operatorname{ctn}(z-\omega_{r,m}),
\tag{12}
$$

where

$$
C_0 = \frac{1}{\pi}\int_{z_0}^{z_0+\pi} f(t)\,dt,
$$

which is given by Briot and Bouquet (loc. cit. p. 291).

Various other forms may be given to formula (10). Thus, since we have the identity

$$
i\operatorname{ctn}(z-a_r) \equiv \frac{2}{1-e^{2ni(z-a_r)}} - 1 = 1 + 2\sum_{n=1}^{\infty}e^{2ni(z-a_r)},
$$

(provided $\eta>\mu_r$, where $z=\xi+i\eta$ and $a_r=\lambda_r+i\mu_r$), and since the sum of the residues is zero, we may write

$$
f(z) = C_0 + 2i\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{A_1^{(r)}}{1-q^{2n}}\left(q^{2n}e^{-2ni(z-a_r)} - e^{2ni(z-a_r)}\right),
\tag{13}
$$

which holds in a strip of the period cell such that $I(z) > I(a_r)$.

In a similar manner, (10) may be written in the form

$$(14)\quad f(z) = C_0 + 2i\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{A_1^{(r)}}{1-q^{2n}}(e^{-2ni(z-a_r)} - q^{2n}e^{2ni(z-a_r)}),$$

which is valid provided $I(z) < I(a_r)$.

5. *Conclusion.* In this concluding section we shall use our formula (9) to obtain the Fourier series expansion of certain elliptic functions. We shall first consider the Weierstrass $\wp$ function, which in the neighborhood of the origin has the expansion

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \cdots.$$

If in (9) we put

$$\alpha = 0,\ \beta = 0,\ m_r = 2,\ A_1 = 0,\ A_2 = 1,\ a_r = 0,$$

we obtain at once

$$(15)\qquad \wp(z, \pi, \pi\tau) = C_0 + \frac{1}{\sin^2 z} - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2nz.$$

The determination of C_0 follows immediately from the fact that

$$\lim_{z\to 0}\left(\wp(z) - \frac{1}{z^2}\right) = 0.$$

Thus, we obtain

$$(16)\qquad C_0 = \frac{1}{3}\left(-1 + 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right) = \frac{1}{3}\,\frac{\vartheta_1'''}{\vartheta_1'}.$$

Next, let us consider the square of the $\wp$ function:

$$\wp^2(z) = \frac{1}{z^4} + \frac{g_2}{10} + c_3z^2 + \cdots.$$

If, in (9), we put

$$A_1 = A_2 = A_3 = 0,\ A_4 = 1;\ \alpha = \beta = 0;\ m_r = 4,\ a_r = 0$$

it will follow, after a slight reduction, that

$$(17)\qquad \wp^2(z, \pi, \pi\tau) = C_0 + \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{16}{3}\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}\cos 2nz.$$

The constant C_0 is now determined by the condition

$$\lim_{z\to 0}\left(\wp^2(z) - \frac{1}{z^4}\right) = \frac{g_2}{10},$$

and the fact that

$$g_2 = \frac{4}{3} + 320\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}},$$

(see Tannery-Molk, vol. 4, p. 107). It is found that

$$(18)\qquad C_0 = \frac{1}{9}\left(1 + 240\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}\right).$$

As a last example we shall consider the square of the Jacobi sine-amplitude function and its reciprocal. In terms of the theta functions, these are essentially equivalent to

$$\frac{\vartheta_2^2\vartheta_3^2\vartheta_1^2(z)}{\vartheta_0^2(z)} \quad\text{and}\quad \frac{\vartheta_2^2\vartheta_3^2\vartheta_0^2(z)}{\vartheta_1^2(z)},$$

respectively. To obtain an expression for the first of these we put, in (9),

$$\alpha = \beta = 0;\ A_1 = 0,\ A_2 = 1;\ a_r = \frac{\pi\tau}{2};\ m_r = 2.$$

The constant C_0 is here determined by the fact that the function vanishes at the origin. After a slight calculation we obtain the expansion

$$(19)\qquad \vartheta_2^2\vartheta_3^2\frac{\vartheta_1^2(z)}{\vartheta_0^2(z)} = 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} + \frac{1}{\sin^2\left(z - \dfrac{\pi\tau}{2}\right)} - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2n\left(z - \frac{\pi\tau}{2}\right).$$

If we replace z by $z+\pi\tau/2$, we obtain

$$(20)\ \vartheta_2^2\vartheta_3^2\frac{\vartheta_0^2(z)}{\vartheta_1^2(z)} = 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} + \frac{1}{\sin^2 z} - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2nz.$$

From (19) we may derive a Fourier series in which the argument is z. Thus, since

$$\frac{1}{\sin^2\left(z-\dfrac{\pi\tau}{2}\right)} = \frac{-4qe^{-2iz}}{(1-qe^{-2iz})^2} = -4\sum_{n=1}^{\infty}nq^ne^{-2niz},$$

provided $\eta<\pi\nu/2$, where $z=\xi+i\eta$ and $\tau=\mu+i\nu$, we find, after some reduction,

$$(21)\qquad \vartheta_2^2\vartheta_3^2\frac{\vartheta_1^2(z)}{\vartheta_0^2(z)} = 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}(1-\cos 2nz),$$

which is valid in a strip bounded by lines through $\pm\pi\tau/2$ and parallel to the real axis.

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