THE PROBABLE ERROR OF CERTAIN FUNCTIONS OF THE ERRORS MADE IN MEASUREMENTS*

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1. *Introduction.* This paper presents with proof and applications several theorems pertaining to the probable error of certain functions of the errors made in measurements. If the measurements or observations are multiplied by a constant b, the probable value of functions of the error of bm will differ from the probable value of these same functions of the error of m the measurement. This article shows that there are values of b such that the probable value of certain functions of the error of bm is less than the probable value of these functions of the error of m.

The probable value of certain functions of the mean are also treated. General frequency laws for the errors are used in the first theorems; these include the discrete, the continuous and a combination of the discrete and the continuous cases. Here Stieltjes integrals are used in the proofs. Special cases are mentioned.

The following theorems are proven by use of general frequency laws which come under the continuous case. The Gaussian law is treated as special cases to the theorems.

2. *Concerning the Square of the Error of the Measurement.*

THEOREM 1. *Let x be the error of the measurement m, d the expected value of x, and c the expected value of x^2. Then under any law of error, whose second moment with respect to the true value a exists, there exist values of the constant b such that the probable value of the square of the error of bm is less than the probable value of the square of the error of the measurement, provided*

$$ad - c \neq 0, \quad a^2 - 2ad + c \neq 0.$$

Under these conditions b lies between

$$1 \quad and \quad 1 - \frac{2(c - ad)}{a^2 - 2ad + c}.$$

* Presented to the Society, April 19, 1930.

PROOF. Let $F(x)$ be the probability that x lies in the interval $(-\infty, x)$ where x is included. Then the probability that x lies in $(-\infty, x^*)$, where x^* means that x has been deleted, is $F(x-0)$. The probability that x lies at x is $F(x+0)-F(x-0)$, the probability that x lies in the segment (x_1^*, x_2^*) is

$$F(x_2 - 0) - F(x_1 + 0) = \int_{x_1+0}^{x_2-0} dF(x).$$

Finally the probability that x lies in the closed interval (x_1, x_2) is

$$F(x_2) - F(x_1 - 0) = \int_{x_1-0}^{x_2} dF(x).$$

We have also $F(-\infty)=0$ and $F(+\infty)=1$. If $f(x)$ is the probability function and is continuous, then

$$F(x) = \int_{-\infty}^{x} f(x)dx. \tag{1}$$

The error of the measurement m from the true value a is $x=a-m$. The error of bm is

$$x' = a - bm = a - b(a - x) = a(1 - b) + bx.$$

The probable value of $(x')^2$ is

$$\left\{\begin{aligned} P(x'^2) &= \int_{-\infty}^{\infty} \left\{a^2(1 - b)^2 + 2a(1 - b)bx + b^2x^2\right\} dF(x) \\ &= a^2(1 - b)^2 + 2abd(1 - b) + b^2c, \end{aligned}\right. \tag{2}$$

which is less than the probable value of the square of the error of m,

$$P(x^2) = \int_{-\infty}^{\infty} x^2 dF(x) = c,$$

if

$$a^2(1 - b)^2 + 2abd(1 - b) + b^2c < c.$$

From this inequality it is seen that b lies between

$$1 \quad \text{and} \quad 1 - \frac{2(c - ad)}{a^2 - 2ad + c}.$$

If the law of error is a Pearson type III,

$$y = k\left(1 + \frac{x}{q}\right)^{hq} e^{-hx},$$

then b lies between

$$1 \text{ and } 1 - \frac{2ke^{hq}(hq - ah + 2)\Gamma(hq + 1)}{a^2h^3(hq)^{hq} - 2akhe^{hq}\Gamma(hq + 1) + ke^{hq}(qh + 2)\Gamma(hq + 1)},$$

provided the denominator is not zero.

COROLLARY. *Under any law of error, whose second moment with respect to the true value a exists and whose first moment with respect to the true value is zero, there exist values of the constant b such that the probable value of the square of the error of bm is less than the probable value of the square of the error of m. These values satisfy the inequalities*

$$1 - \frac{2c}{a^2 + c} < b < 1.$$

The laws of error considered in the corollary include those which are symmetrical with respect to the true value and also others which are skew.

The preceding theorem does not apply to the law of error

$$f(x) = \frac{h}{\pi(1 + h^2x^2)},$$

toı the second moment does not exist.

If the law of error for the error of m is

$$f(x) = \frac{2h}{\pi(1 + h^2x^2)^2}, \tag{3}$$

then b satisfies the inequalities

$$1 - \frac{2}{a^2h^2 + 1} < b < 1.$$

If the error law is a Pearson type X†,

$$f(x) = \frac{h}{2}e^{-h|x|}, \tag{4}$$

then b satisfies the inequalities

† Elderton, *Frequency Curves and Correlation.*

$$1 - \frac{4}{a^2h^2 + 2} < b < 1.$$

If the error law is the Gaussian law,

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2},$$

then the corollary to Theorem 1 becomes in this special case identical to a theorem given by Dodd.*

If the error law for the errors of the measurement m is $f(x) = 1/(2k)$ in $(-k, k)$ and zero elsewhere, then

$$1 - \frac{2k^2}{3a^2 + k^2} < b < 1.$$

This last law is used for the law of errors which are made in using a table of logarithms, for here the error is constant in a certain interval and zero elsewhere.

3. *Concerning the Error of the Mean.* If M is the mean and B is a constant it is desirable to know the nature of the probable value of certain functions of the error of BM and also the probable value of these same functions of the error of M. The following theorems will be useful in this respect.

THEOREM 2. *If the error law for the individual variable is of limited variation and symmetrical with respect to the true value, then the error law for the mean is also symmetrical with respect to the true value.*†

PROOF. Let $f(x)dx$ be the probability that the error lies in the interval $(x, x+dx)$. The probability that the sum lies in the interval $(u, u+du)$ is, according to a theorem given by Dodd,‡

$$p_n(u) = \frac{1}{\pi}\int_0^\infty g(t) \cos (ut)dt,$$

* Dodd, *The error-risk of certain functions of the measurements*, Monatshefte für Mathematik und Physik, vol. 24 (1913), pp. 268–276; Dodd, *The probability of the arithmetic mean compared with that of certain other functions of the measurements*, Annals of Mathematics, (2), vol. 14 (1912–13), pp. 186–198.

† Theorem 2 can be proved for the discrete case.

‡ Dodd, *The frequency law of a function of variables with given frequency laws*, Annals of Mathematics, (2), vol. 27 (1925–26), pp. 12–20.

where the parameter u does not enter the function $g(t)$. The function $p_n(u)$ is an even function and hence is symmetrical with respect to the origin which has been chosen as the true value. The law for the mean is obtained by replacing $p_n(u)$ by $p_n(nu)n$, and this is symmetrical to the true value.

In the theory of sampling from a parent population which is known, the standard error of the mean is equal to the standard error of the parent population divided by the square root of the number of the variates in the sample, provided the parent contains an infinite number of variates.* If the true value of the quantity to be measured is a and is considered to be the mean of the parent population the standard error of the mean can in general be found for the mean from that of the error law for the individual measurements. This idea is treated also in the theory of expected values leading up to the Tchebycheff inequalities.†

When the error laws for the individual errors of the measurements are the same, then the expected value of the mean of the errors is the same as the expected value of an individual error. This can be treated also by sampling theory.

THEOREM 3. *Let x be the error of the measurement m, d the expected value of x and c the expected value of x^2. Then under any law of error whose second moment with respect to the true value a exists, there exist values of the constant B such that the probable value of the square of the error of BM is less than the probable value of the square of the error of M, provided*

$$adn - c \neq 0, \quad a^2n - 2adn + c \neq 0.$$

Under these conditions B lies between

$$1 \quad and \quad 1 - \frac{2(c - adn)}{a^2n - 2adn + c}.$$

The proof of this theorem is similar to that of Theorem 1.

COROLLARY. *Under any law of error, whose second moment exists and whose first moment with respect to the true value a is zero, there exist values of the constant B such that the probable value*

* C. H. Richardson, *The Statistics of Sampling*, a dissertation for the doctorate at The University of Michigan.

† Fisher, *The Mathematical Theory of Probability*, pp. 104–109.

of the square of the error of BM is less than the probable value of the square of the error of M; these values satisfy the inequalities

$$1 - \frac{2c}{na^2 + c} < B < 1,$$

where M is the mean of n measurements and c is the expected value of the square of the error.

The proof of this corollary is similar to that of the Corollary to Theorem 1, since Theorem 2 shows that the error law for the mean is such that its first moment with respect to the true value is also zero.

If the law for the individual errors is (3), then the law for the mean of the errors is*

$$\frac{1}{\pi(hn)^n} \sum_{r=1}^{n+1} \binom{n}{r-1} \frac{\Gamma(r) \cos (r \cdot \tan^{-1} u)}{(hn)^{-n-1}(n^2 + n^2h^2u^2)^{r/2}}.$$

In this case the constant B satisfies the following inequalities:†

$$1 - \frac{2}{na^2h^2 + 1} < B < 1,$$

where n measurements are made.

If the error law for the individual measurement is (4), then the law for the mean is ‡

$$nhe^{-nh|u|} \sum_{r=0}^{n-1} \frac{n^{(-r)} hn \left| u \right|^{n-1-r}}{2^r(n - 1 - r)!(r)!}.$$

In this case

$$1 - \frac{4}{na^2h^2 + 2} < B < 1.$$

If the error law for the errors of the measurement m is $f(x) = 1/(2k)$, then

* Baten, *Theorems concerning probability*, a dissertation for the doctorate at the University of Michigan, 1929. Karl Mayr, *Wahrscheinlichkeitsfunktionen und ihre Anwendungen*, Monatshefte für Mathematik und Physik, vol. 30 (1920), pp. 17–44.

† These inequalities for B can be obtained most readily from the three inequalities for b at the end of §2 by replacing h^2 by nh^2; and the validity of this process follows from the second paragraph above Theorem 3.

‡ See Baten, loc. cit.

$$1-\frac{2k^2}{3na^2+k^2}<B<1.$$

If the error law is

$$f(x)=\frac{2h}{\pi(e^{hx}+e^{-hx})},$$

then

$$1-\frac{2\pi^2}{4a^2h^2+\pi^2}<b<1,$$

and

$$1-\frac{2\pi^2}{4na^2h^2+\pi^2}<B<1.$$

If the error function is

$$f(x)=\frac{ph}{2\Gamma(1/p)}e^{-hp|x|^p},$$

we shall have

$$1-\frac{2\Gamma(3/p)}{\Gamma(1/p)na^2h^2+\Gamma(3/p)}<b<1,$$

and

$$1-\frac{2\Gamma(3/p)}{\Gamma(1/p)na^2h^2+\Gamma(3/p)}<B<1.$$

Other error laws may be used.

4. *Concerning the Absolute Error of the Measurement.*

THEOREM 4. *Under any law of error which is symmetrical with respect to the true value a, whose first moment with respect to the true value exists and which has only one maximum which is at the true value, there exist values of the constant b such that the probable value of the absolute error of bm is less than the probable value of the absolute error of the measurement m. A sufficient condition for such a value is*

$$1-\frac{r}{ka^2+r}<b<1,$$

where r is one-half of the mean deviation and k is the maximum value, and a is assumed to be positive.

PROOF. The error $|x'|=|a-bm|=|a(1-b)+bx|$, where x is the error of the measurement m. The probable value of $|x'|$ is

$$P(|x'|) = \int_{-\infty}^{\infty} |a(1-b)+bx| f(x)dx,$$

where $f(x)$ is the error law. The quantity $\{a(1-b)+bx\}$ is negative when

$$x < -\frac{a(1-b)}{b} = -R.$$

Hence

$$\begin{aligned}
P(|x'|) &= -\int_{-\infty}^{-R}\{a(1-b)+bx\}f(x)dx \\
&\quad + \int_{-R}^{\infty}\{a(1-b)+bx\}f(x)dx \\
&= a(1-b)\left\{-\int_{-\infty}^{-R} f(x)dx + \int_{R}^{\infty} f(x)dx\right\} \\
&\quad + a(1-b)\int_{-R}^{R} f(x)dx + b\left\{-\int_{-\infty}^{-R} xf(x)dx\right. \\
&\quad \left. + \int_{R}^{\infty} xf(x)dx\right\} + b\int_{-R}^{R} xf(x)dx \\
&= a(1-b)\int_{-R}^{R} f(x)dx + 2b\int_{R}^{\infty} xf(x)dx \\
&< a(1-b)2ka(1-b)/b + 2b\int_{0}^{\infty} xf(x)dx,
\end{aligned}$$

which is less than

$$P(|x|) = \int_{-\infty}^{\infty} |x| f(x)dx,$$

if

$$2a^2k(1-b)^2/b + 2br < 2\int_0^{\infty} xf(x)dx,$$

or if

$$2a^2k(1-b)^2 + 2b^2r - 2br < 0,$$

or if

$$1 - \frac{r}{a^2k+r} < b < 1.$$

If the error law is the Gaussian law then the above result is identical to that obtained by Dodd who used altogether the Gaussian law.

THEOREM 5. *Under the law* $f(x) = he^{-h|x|}/2$, *there exist values of the constant* b *such that the probable value of the absolute error of* bm *is less than the probable value of the absolute error of the measurement* m. *A sufficient condition for such a value is*

$$1 - \frac{1 - ah}{a^2h^2 - ah + 1} < b < 1,$$

where a *is positive.*

PROOF. By a similar process,

$$P(|x'|) = ha(1 - b)\int_0^R e^{-hx}dx + hb\int_R^\infty xe^{-hx}dx$$

$$= a(1 - b)(1 - e^{-hR}) + bRe^{-hR} + be^{-hR}/h.$$

Since

$$e^{hR} = (1 + hR + h^2R^2/2 + \cdots),$$

we have $e^{hR} > hR$ and $e^{-hR} < 1/(hR)$. We have also

$$1 - e^{-hR} = \left(1 - 1 + hR - \frac{h^2R^2}{2!} + \frac{h^3R^3}{3!} - \cdots\right)$$

$$= hR - \left(\frac{h^2R^2}{2!} - \frac{h^3R^3}{3!} + \cdots\right).$$

The last parenthesis is positive, therefore

$$1 - e^{-hR} < hR = a(1 - b)h/b.$$

Therefore

$$P(|x'|) \leqq ha^2(1 - b)^2/b + hR + b/h,$$

which is less than $P(|x|) = 1/h$ if

$$h^2a^2 - 2a^2h^2b + a^2h^2b^2 + hba(1 - b) + b^2 < b,$$

or

$$1 - \frac{1 - ah}{a^2h^2 - ah + 1} < b < 1.$$

The denominator is not zero unless ah, a, or h is imaginary. These cases are excluded here. If $1 > ah$ the fraction in the first inequality is positive. For other cases the inequalities have no meaning. Theorem 4 gives a lower limit for b regardless of the true value of a.

If the error law is

$$f(x) = \frac{2h}{\pi(1 + h^2x^2)^2},$$

we have

$$1 - \frac{1}{2a^2h^2 + 1} < b < 1.$$

Under the law

$$f(x) = \left(\frac{h}{4} + \frac{h^2|x|}{4}\right) e^{-h|x|},$$

we have

$$1 - \frac{3}{a^2h^2 + 3} < b < 1.$$

Under a type of Bayes function

$$f(x) = \frac{3h(1 - h^2x^2)}{4},$$

we have

$$1 - \frac{1}{4a^2h^2 + 1} < b < 1.$$

If

$$f(x) = \frac{2h}{\pi(e^{hx} + e^{-hx})},$$

we have

$$1 - \frac{\sum_0^\infty(-1)^n/(2n+1)^2}{ha^2 + \sum_0^\infty(-1)^n/(2n+1)^2} < b < 1.$$

Under the error law

$$f(x) = \frac{hp}{2\Gamma(1/p)} e^{-hp|x|^p},$$

we have

$$1 - \frac{\Gamma(2/p)}{ph^2a^2 + \Gamma(2/p)} < b < 1.$$

Under the law $f(x) = 1/(2k)$, where k is a constant, we have

$$1 - \frac{k^2}{2a^2 + k^2} < b < 1.$$

5. *Concerning the Absolute Value of the Error of the Mean.* If the Gaussian law is the error law for the individual errors, then the law for the mean of n is also another Gaussian law. According to Theorem 4, we may write

$$1 - \frac{1}{2h^2a^2\sqrt{n}+1} < B < 1, \tag{5}$$

where B is a constant which is multiplied by the mean M. This indicates that the mean deviation of the mean is larger than the mean deviation of B times the mean, provided the law of error is the Gaussian law. This indicates that there is a better value than the mean for the true value, if the law is the Gaussian law.

The method of random sampling from a known parent population may be employed to investigate the nature of B for other laws of error. Let the parent population consist of the variates $x_1, x_2, x_3, \cdots, x_s$. Let samples of r variates be taken from this parent. It has been shown† that the mean of all possible sample means is equal to the mean of the parent population, if repetitions of samples are not allowed.

Let the first sample mean of r variates be

$$Z_1 = \frac{x_1 + x_2 + x_3 + \cdots + x_r}{r},$$

then the deviation of the first sample mean from the mean of the sample means is, in absolute value,

$$\begin{aligned} |\bar{Z}_1| &= \left| \frac{x_1 + x_2 + x_3 + \cdots + x_r}{r} - M_x \right| \\ &= \left| \frac{(x_1 - M_x) + (x_2 - M_x) + (x_3 - M_x) + \cdots + (x_r - M_x)}{r} \right| \\ &= \left| \frac{\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \cdots + \bar{x}_r}{r} \right| \\ &\leqq \frac{|\bar{x}_1| + |\bar{x}_2| + |\bar{x}_3| + \cdots + |\bar{x}_r}{r}, \qquad \bar{x}_i = x_i - M_i, \end{aligned}$$

where the mean of the parent population is M_x.

† An editorial, Annals of Mathematical Statistics, vol. 1 (1930), pp. 101–121. See also the first footnote on page 61.

The mean deviation of the sample means is found by finding the average of all possible $|\overline{Z}|$'s and this is clearly less than or equal to the mean deviation of the parent population. The mean deviation of the sample means is

$$\frac{\sum_{i=1}^{{}_sC_r} |\overline{Z}_i|}{{}_sC_r},$$

while the mean deviation of B times the sample means is

$$\frac{\sum_{i=1}^{{}_sC_r} B|\overline{Z}_i|}{{}_sC_r} = \frac{B\sum_{i=1}^{{}_sC_r} |\overline{Z}_i|}{{}_sC_r},$$

where the summations are divided by ${}_sC_r$ since there are that many possible samples that can be taken from the s variates. The mean deviation of B times the sample means is less than the mean deviation of the sample means if B is less than unity. If s goes to infinity the above is also true.

The above inequalities for the Gaussian law give more information than the above sampling method. B is not only less than 1 but it lies in a certain interval with the end points deleted. This is found in (5). Going back to sampling, if the mean of the parent population is the true value then the mean deviation of B times the sample means is less than the mean deviation of the mean of the sample means.

To be able to find the inequalities for B when other laws of error are used it is necessary to find the law for the mean when the law for the individual error law is known.

The limits for b are not always the best limits, for under Theorem 1 the lower limit for b is found by setting the first derivative of (2) equal to zero and solving for b. This gives the middle of the interval as the lower limit of b, which brings b nearer to unity.

Intervals for the constant b can be found when the higher moments are used.

In general the coefficients b and B are so near to unity that bm and BM do not differ appreciably from m and M respectively, yet this is not always the case, notably, when the standard error of m is large relative to m.

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