above involution to the entire family of penosculating conics. If, further, we define the cross-ratio of four members of a one-parameter family of conics to be that of the parameters which single them out in the family, we can conclude that the pairs of conics of equal eccentricity separate harmonically the osculating equilateral hyperbola and the ellipse of minimum eccentricity. In this involution of penosculating conics so determined the partner of the osculating parabola is the degenerate parabola consisting of the tangent to the given curve counted twice. Although as Wilczynski found, there is in this family a unique ellipse of minimum eccentricity, there is no hyperbola of maximum eccentricity. The osculating equilateral hyperbola and the ellipse of minimum eccentricity appear as the double conics in this involution of penosculating conics.

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THE BITANGENTIAL CURVE*

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1. *Introduction.* The bitangential curve of an algebraic surface is the locus of the points of contact of planes which touch the surface in two points. It is the reciprocal of the nodal developable. The order of this curve, for an algebraic surface without singularities of any kind, has been determined by Cayley,† who called it the node-couple curve.

When the surface has a nodal and a cuspidal curve, each of given order, the orders of the spinodal and flecnodal curves were found by Cayley.‡ The order of the bitangential curve of a surface with nodal and cuspidal curves, however, has not been found explicitly. Basset§ makes the following statement: "I have not succeeded in ascertaining the reduction in the degree of the bitangential curve which is produced by a nodal and a cuspidal curve; but if the reduction is denoted by $xb+yc$, the method of the preceding paragraph indicates that x and y are

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† A. Cayley, Collected Mathematical Papers, vol. 6, p. 346.

‡ A. Cayley, loc. cit., vol. 6, pp. 342–343.

§ A. B. Basset, *A Treatise on the Geometry of Surfaces*, Cambridge, 1910, p. 280.

functions of the degree n of the surface," which, together with a similar statement in §59, page 40, indicates that the order of the bitangential curve under these conditions has not been determined.

The purpose of this paper is to determine the order of the bitangential curve of an algebraic surface that has nodal and cuspidal curves and also singular points and planes. The necessary formulas for this purpose have all been derived by Cayley and Zeuthen, so that the work in this paper consists merely in collecting and combining these formulas.

2. *Notation.* The following symbols will be used:

$n[n']$ order [class] of algebraic surface f.

a' class of plane section of f.

κ' number of inflections of plane section of f.

$b[c]$ order of nodal [cuspidal] curve of f.

$k[h]$ number of apparent double points of nodal [cuspidal] curve.

t number of triple points of nodal curve.

$\beta[\gamma]\{i\}$ number of intersections of nodal and cuspidal curves which are cusps on cuspidal [nodal] {neither} curve.

$\rho'[\sigma']$ order of bitangential [spinodal] curve of f.

The above notation is that used by Cayley and Salmon. The complete list (given in the references*) of symbols representing all the possible singularities of an algebraic surface is very long and in a short paper, it seems unnecessary to define all of them.

3. *Order of the Bitangential Curve.* The following formulas derived by Cayley† hold for an algebraic surface with no further singularities than a nodal and a cuspidal curve:

$$a'(n' - 2) = \kappa' - \rho' + 2\sigma', \tag{1}$$

$$a' = n(n - 1) - 2b - 3c, \tag{2}$$

$$\kappa' = 3n(n - 2) - 6b - 8c, \tag{3}$$

* Salmon-Rogers, *A Treatise on the Analytical Geometry of Three Dimensions*, 5th ed., 1915, vol. 2, pp. 314–318. H. G. Zeuthen, *Sur la théorie des surfaces réciproques*, Mathematische Annalen, vol. 10 (1876), pp. 450–454.

† A. Cayley, loc. cit., vol. 11, pp. 228–231.

$$(4) \qquad n' = n(n-1)^2 - n(7b+12c) + 4b^2 + 8b + 9c^2 + 15c - 8k - 18h - 9t + 18\beta + 12\gamma + 12i,$$

$$(5) \qquad \sigma' = 4n(n-2) - 8b - 11c.$$

Solving (1) for ρ' and substituting in it the values of a', κ', n', σ' as given in formulas (2), (3), (4), (5), we find

$$\begin{aligned}\rho' = {} & n(n-2)(n^3 - n^2 + n - 12) - n(n-1)[3n(3b+5c) \\ & - 2b(2b+5) - 9c(c+2)] + n(2b+3c)(7b+12c) \\ & - (2b+3c)[4(b-1)(b+3) + 3c(3c+5)] + 2b \\ & - [n(n-1) - 2b - 3c][18(h-\beta) - 12(i+\gamma) + 8k + 9t].\end{aligned}$$

This is the order of the bitangential curve when the surface has a nodal curve of order b and a cuspidal curve of order c. The first product is the order of the bitangential curve for a non-singular surface. The reduction in the order of the bitangential curve due to the nodal and cuspidal curves is the sum (with the signs changed) of all the terms of the above expression after the first product.

The order can also be found by the same method when the surface has any of the point or plane singularities defined in the complete list of singularities. In this case, numbers associated with certain of these singularities occur in formulas (1), (4) and (5) as determined by Zeuthen in the reference cited. The formula for ρ' is very long when all possible singularities are included and, therefore, will not be given here. The reduction in the order of the bitangential curve due to any given singularity of the surface can be readily determined, however, as above.

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