ON A FUNCTION CONNECTED WITH A CUBIC FIELD*

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By examining the functional equation of the zeta function and similar functions of an algebraic field, Hecke‡ has indicated a mode of constructing three types of modular forms. The first two types, associated with the rational Dirichlet L-functions and the Dedekind zeta function in an imaginary quadratic field, respectively, were already known. The third type, associated with a real quadratic field, was new.

Following Hecke, we construct in this note a certain function associated with a cubic field of negative discriminant. Let K denote such a field, and let $\zeta_K(s)$ be the zeta function in this field. Then as Artin§ has shown, $\zeta_K(s)/\zeta(s)$, the quotient of this function by the Riemann zeta, is an entire function of s. Indeed

$$\zeta_K = \zeta(L_1 L_2)^{1/2}, \tag{1}$$

where L_1 and L_2 are L-functions in the imaginary quadratic field generated by the square root of d, the discriminant of K.

If we define $G(n)$ by

$$\frac{\zeta_K}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{G(n)}{n^s}, \quad \sigma = R(s) > 1,$$

then the function we shall consider may be exhibited as

$$M(y) = \sum_{1}^{\infty} G(n) e^{2n\pi y i/\Delta},$$

$$I(y) > 0, \ \Delta = |d|^{1/2} > 0.$$

Using a well known formula of Mellin,||

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‡ *Zur Theorie der elliptischen Modulfunktionen*, Mathematische Annalen, vol. 97 (1926), pp. 210–242.

§ *Über die Zetafunktionen gewisser algebräischen Zahlkörper*, Mathematische Annalen, vol. 89 (1923), pp. 147–156.

|| Mellin, *Abriss einer einheitlichen Theorie der Gamma- und hypergeometrischen Funktionen*, Mathematische Annalen, vol. 68 (1910), pp. 305–337.

$$e^{2n\pi yi/\Delta} = \frac{1}{2\pi i}\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{-2n\pi yi}{\Delta}\right)^{-s}\Gamma(s)ds,$$

$$(2)\quad \begin{cases} M(y) = \dfrac{1}{2\pi i}\displaystyle\sum_{1}^{\infty}\int\left(\frac{-2\pi yi}{\Delta}\right)^{-s}\Gamma(s)\frac{G(n)}{n^s}ds \\ \quad = \dfrac{1}{2\pi i}\displaystyle\int\left(\frac{-2\pi yi}{\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds, \end{cases}$$

the interchange of integration and summation is easily justified. We now consider the integral of

$$\left(\frac{-2\pi yi}{\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)$$

taken in the positive direction around the rectangle of vertices

$$\frac{3}{2} \pm Ai, \quad -\frac{1}{2} \pm Ai, \quad A > 0.$$

Since ζ_K/ζ is integral and since ζ_K has a zero* at $s=0$, the integrand is regular within and on the boundary of the rectangle. Furthermore, using (1) and†

$$L_i(s) = O(t^2), \qquad (i = 1, 2),$$

$$s = \sigma + it, \quad \sigma \geqq -\frac{1}{2};$$

$$\frac{\zeta_K}{\zeta}(s) = O(t^2) \text{ for } \sigma \geqq -\frac{1}{2};$$

we see that the integrals along the vertical boundaries will converge and those along the horizontal boundaries will approach zero when A becomes infinite.‡ Therefore

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* Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, 1918, Satz 155.

† Landau, *Ueber Ideale und Primideale in Idealklassen*, Mathematische Zeitschrift, vol. 2 (1918), p. 106.

‡ See Landau, *Vorlesungen über Zahlentheorie*, vol. 1, 1927, p. 215.

$$\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{-2\pi y i}{\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds$$

$$=\int_{-1/2-\infty i}^{-1/2+\infty i}\left(\frac{-2\pi y i}{\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds$$

$$=\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{-2\pi y i}{\Delta}\right)^{s-1}\Gamma(1-s)\frac{\zeta_K}{\zeta}(1-s)ds. \tag{3}$$

Now

$$\zeta_K(1-s)=\left(\frac{\Delta}{2\pi^{3/2}}\right)^{2s-1}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma(s)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma(1-s)}\zeta_K(s),$$

and

$$\zeta(1-s)=\pi^{-(2s-1)/2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s),$$

so that

$$\frac{\zeta_K}{\zeta}(1-s)=\left(\frac{\Delta}{2\pi}\right)^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}\frac{\zeta_K}{\zeta}(s).$$

Substituting in (3), we find that

$$M(y)=\frac{1}{2\pi i}\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{-2\pi y i}{\Delta}\right)^{s-1}\left(\frac{\Delta}{2\pi}\right)^{2s-1}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds$$

$$=\frac{1}{-yi}\frac{1}{2\pi i}\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{2\pi i}{y\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds.$$

Comparing this with (2), we see at once that

$$M(y)=-\frac{1}{yi}M\left(-\frac{1}{y}\right),$$

which is the property of $M(y)$ sought.

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