be set up by combining the integral around the boundary with integrals over specified curves interior to the region, or by using area integrals together with line integrals, or by admitting values of the error at isolated points in the expression to be minimized. It is clear that this type of generalization would lead ultimately to the consideration of a Stieltjes integral, though the precise degree of generality that would be practicable remains to be ascertained.

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CONCERNING QUASI-k-FOLD TRANSITIVITY OF PERMUTATION GROUPS*

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1. *Introduction.* Let G denote a permutation group having the property that for every l such that $1 \leqq l \leqq k$ it is true that when two sets of l letters each are given, taken from the letters on which G operates, then there exists in G a permutation P which transforms the first of these sets of letters in some order into the second. Then it will be said that G is quasi-k-fold transitive.† It is clear that G is transitive in the ordinary sense. Quasi-k-fold transitivity differs from k-fold transitivity in respect to the matter of order in the elements; in the latter a permutation P exists when the order of the l elements in each of the two sets is prescribed such that it transforms the one ordered set into the other; in the former we have to do with the transformations of unordered sets.

G. A. Miller‡ has pointed out that, when p is a prime number of the form $4x+3$, the semi-metacyclic group of degree p is quasi-2-fold transitive, in the sense of our definition, even though it is only singly transitive. In a paper not yet published Miller has easily proved that a quasi-k-fold transitive group is

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† Since the text of this article was put into type Professor W. B. Carver has called my attention to a paper by W. B. Carver and Mrs. Estella Fisher King, this Bulletin, vol. 26 (1920), pp. 319–322, dealing with quasi-k-fold transitivity.

‡ Transactions of this Society, vol. 28 (1926), p. 339.

primitive except possibly when k is 1. Though the concept of quasi-k-fold transitivity seems to be one of some interest I have found no other reference to it than what has just been mentioned.

In the present paper I prove (§2) one general theorem concerning the order of these groups; I exhibit (§3) an infinite class of doubly (but not triply) transitive groups which are quasi-3-fold transitive and show (§4) that each group in this class contains a singly transitive subgroup which is quasi-2-fold transitive, the latter groups containing among them the semi-metacyclic groups already mentioned. Moreover I give (§5) two examples of 3-fold transitive groups which are quasi-4-fold transitive and point out (§6) some additional properties of one of these groups.

2. *Order of the Groups.* Let G denote a quasi-k-fold transitive group. Let m be the order of the largest subgroup H of G each element of which leaves fixed each of k letters of G and let K be the largest subgroup of G which permutes these k letters among themselves. Then K contains H as a self-conjugate subgroup. Let μm denote the order of K. Then the named k elements are permuted among themselves by K according to a group of order μ, whence it follows that μ is a factor of $k!$. If G is k-fold transitive then $\mu = k!$.

If the degree of G is n then $n(n-1)\cdots(n-k+1)/(k!)$ is the number of sets of k letters each which may be formed from the letters of G. From this it follows that the order of G is $n(n-1)\cdots(n-k+1)\mu m/(k!)$.

We are thus led to the following theorem:

The order of a quasi-k-fold transitive group G of degree n is $n(n-1)\cdots(n-k+1)\mu m/(k!)$, where m is the order of the largest subgroup H of G which leaves unchanged each of k given letters of G and where μ is the order of the group by which these k letters are permuted under the largest subgroup K of G which permutes these letters among themselves. The order of K is μm. The subgroup H is contained self-conjugately in the subgroup K.

When G is k-fold transitive in the ordinary sense this result reduces to a classic theorem concerning multiply transitive groups.

3. *An Infinite Class of Doubly Transitive Groups.* Let p be

an odd prime number and let us consider the group G composed of the transformations

$$x' = \frac{ax + b}{cx + d}$$

where a, b, c, d are marks of the Galois field $GF[p^n]$ such that $ad-bc$ is a square in $GF[p^n]$. Then G is doubly (but not triply) transitive when it is represented as a simply isomorphic permutation group on the p^n+1 symbols consisting of ∞ and the marks of $GF[p^n]$; its order is $\frac{1}{2}(p^n+1)p^n(p^n-1)$. We shall show that *this permutation group G is quasi-3-fold transitive when p^n is of the form $4x+3$, that is, when p is of the form $4x+3$ and n is odd.*

For this purpose it is evidently sufficient to show that G contains a permutation which transforms any three preassigned marks u, v, w whatever in some order into 0, 1, ∞. Since G is doubly transitive it contains a permutation changing u, v into 0, ∞; let t be the mark into which w is changed by this transformation. Then if t is a square the group G contains the transformation $x'=t^{-1}x$ and this changes t to 1 while 0 and ∞ are left fixed, so that in this case u, v, w are changed to 0, ∞, 1 respectively. In case t is not a square we must use a different transformation. But in this case $-t$ is a square, since -1 is a not-square owing to the fact that p^n is of the form $4x+3$. We then use the transformation $x'=t/x$ to change t into 1; it interchanges 0 and ∞. Hence, in this case, u, v, w are changed into ∞, 0, 1 respectively.

Now let d be any factor of n and write $n=d\delta$. Adjoin to G the transformation

$$x' = x^{p^\delta}$$

and thus construct an enlarged group G_d (with $G_1 \equiv G$). The latter group is doubly (but not triply) transitive and contains G as a subgroup of index d. It is now obvious that G_d is quasi-3-fold transitive when p^n is of the form $4x+3$.

If any positive integer L is assigned in advance then odd integers n exist having more than L factors d, whence it follows that when L is given there exists an infinitude of prime powers p^n such that for any one of these there exist more than L quasi-3-fold doubly (but not triply) transitive groups of degree p^n+1.

4. *Relations between Groups. Each of the quasi-3-fold doubly (but not triply) transitive groups G_d of degree p^n+1, described in §3, contains as a subgroup of index p^n+1 a quasi-2-fold transitive subgroup of degree p^n whose degree of transitivity is unity.* It is evidently sufficient to prove this for the case of the named subgroup of G, ∞ being the element held fixed in forming the subgroup. The transformations of this subgroup are those of the form $x'=a^2x+b$, where a and b are marks of the $GF[p^n]$ and the symbols permuted are the marks of this field. It is sufficient to show that G contains a permutation transforming any two marks u, v whatever in some order into the pair 0, 1. This group contains the transformations $x'=x-u$ and $x'=x-v$; these replace u and v by 0, $v-u$ and $u-v$, 0 respectively. One of the marks $v-u$ and $u-v$ is a square ρ^2 since -1 is a not-square when p^n is of the form $4x+3$ (as now supposed): the mark ρ^2 is changed to 1 by the transformation $x'=\rho^{-2}x$ and this transformation belongs to the group; it leaves 0 fixed. Therefore we conclude to the proposition asserted.

5. *Examples of 3-Fold Transitive Groups.* It is well known that the group consisting of all the transformations

$$x' = \frac{ax^{p^t} + b}{cx^{p^t} + d},$$

where a, b, c, d are marks of $GF[p^n]$ such that $ad-bc\neq 0$ and t ranges over the set $0, 1, \cdots, n-1$, permutes ∞ and the marks of $GF[p^n]$ according to a triply transitive group of degree p^n+1 and order $(p^n+1)p^n(p^n-1)n$. Let us ask for the conditions under which this group is quasi-4-fold transitive. From the theorem in §2 it follows that we must then have the equation

$$(p^n + 1)p^n(p^n - 1)n = \frac{1}{24}(p^n + 1)p^n(p^n - 1)(p^n - 2)\mu m,$$

where μ and m are integers and μ is a factor of 24. Therefore we must have

$$(p^n - 2)\mu m = 24n \text{ and } p^n \leqq 24n + 2.$$

In order to satisfy the given inequality we must have $p\leqq 26$, $n\leqq 7$, with further restrictions on n for the larger values of p. Then it is easy to show that the given equation can be satisfied

only when $p=2$, $n=2$, 3, 5; $p=3$, $n=1$; $p=5$, $n=1$. When $p=3$, $n=1$, we have the (four-fold transitive) symmetric group of degree 4. When $p=5$, $n=1$, the group is of order $6\cdot5\cdot4$ and is not quasi-4-fold transitive. When $p=2$, $n=2$, we have the (five-fold transitive) symmetric group of degree 5.

When $p=2$, $n=3$, we have a triply transitive group G of degree 9 and order $9\cdot8\cdot7\cdot3$. If ω is a primitive mark of $GF[2^3]$ satisfying the equation $\omega^3=\omega+1$ then the symbols ∞, 0, 1, ω are permuted among themselves according to a group of order 12 by the group whose generators are $x'=\omega/x$ and $x'=\omega^3x^4+1$, and these symbols are permuted among themselves by no larger subgroup of G. Hence G transforms the four symbols into $9\cdot8\cdot7\cdot3/12$, or 126, quadruples. But there are just 126 quadruples of nine things. Hence the named group G is quasi-4-fold transitive.

There is left for consideration the case $p=2$, $n=5$. The group G is then of degree 33 and order $33\cdot32\cdot31\cdot5$. Let ω be a primitive mark of $GF[2^5]$ satisfying the equation $\omega^5=\omega^2+1$. The symbols ∞, 0, 1, ω are permuted among themselves according to a group of order 4 by the group whose generators are $x'=\omega/x$ and $x'=(x+\omega)/(x+1)$. The transformations $x'=1/x$, $x'=x+1$, $x'=x^2$ permute ∞, 0, 1 among themselves and generate a group H which is transitive on the remaining 30 symbols. Since G is triply transitive it has an element transforming any four symbols α, β, γ, δ into ∞, 0, 1, t, where t is some mark of $GF[2^5]$; using H this set can be changed in some order to ∞, 0, 1, ω. Since any set α, β, γ, δ of four symbols can be transformed into the set ∞, 0, 1, ω it follows that G is quasi-4-fold transitive.

6. *Additional Properties.* It is of interest to note some additional properties of the named group G of degree 33 and order $33\cdot32\cdot31\cdot5$. It may readily be shown that the symbols ∞, 0, 1, ω, ω^{18} are permuted among themselves by just four elements of G and hence that this set is transformed into just $33\cdot8\cdot31\cdot5$ quintuples by the permutations induced by G. Moreover it can readily be shown that any two sets of four symbols each occur equally often in these quintuples and hence that each quadruple appears in just five of the quintuples. From this it follows that a given quadruple occurs in the quintuples with just five other elements; these elements may therefore be taken together to

form a new quintuple. The quintuples which may be formed in this way constitute the same set of quintuples as that already formed. Therefore this set of quintuples constitutes a rather remarkable tactical configuration.

Since with each set of four symbols we have, as just indicated, a unique associated set of five symbols, we may form a configuration of nines by taking with each four symbols the five which are thus associated with it. Thus we have a configuration of $33 \cdot 8 \cdot 31 \cdot 5$ nines, such that each set of four symbols appears just 126 times, each triple just 630 times and each pair just 2790 times.

It may be shown that the eleven symbols,

$$\infty, 0, 1, \omega, \omega^2, \omega^5, \omega^{11}, \omega^{20}, \omega^{26}, \omega^{29}, \omega^{30}$$

are permuted among themselves by a subgroup of G of order 110 and by no larger subgroup, whence it follows that this set is transformed by G into $31 \cdot 3 \cdot 16$ sets of eleven each. Let A denote the tactical configuration constituted by these sets of eleven symbols each. It may be shown that this configuration has the following properties: each set of four symbols occurs in twelve and just twelve of these sets of eleven each; each triple of the 33 symbols appears in just 45 of the sets.

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