NOTE ON PERIODIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES*

BY W. F. OSGOOD

Let $f_1(x), f_2(x), \cdots, f_m(x)$ be periodic functions of the complex variable x, each meromorphic in its fundamental domain of periodicity $\mathfrak{F}$ (parallelogram or closed strip of periods). Let each function admit the period or periods corresponding to $\mathfrak{F}$, and let there be no other periods common to all the functions except such as are derived linearly and integrally from those of $\mathfrak{F}$. Then a suitable linear combination of the above functions,

$$C_1 f_1(x) + \cdots + C_m f_m(x),$$

will admit the periods corresponding to $\mathfrak{F}$, and no others.

The corresponding theorem is not true for periodic functions of several complex variables. The proof is given by the following example. Let

$$F(u_1, u_2, u_3) = u_1 - \zeta(u_3),$$

$$\Phi(u_1, u_2, u_3) = u_2 - \zeta(u_3),$$

where

$$\zeta(z) = \frac{d}{dz} \log \sigma(z) = \frac{\sigma'(z)}{\sigma(z)},$$

and $\sigma(z)$ is the Weierstrassian sigma-function. Here

$$\zeta(z + \omega_1) = \zeta(z) + \eta_1,$$

$$\zeta(z + \omega_2) = \zeta(z) + \eta_2,$$

where ω_1, ω_2, η_1, η_2 are connected by Legendre's relation,

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i.$$

These functions F and Φ obviously admit the periods

---

* Read before the mathematical group at the University of California at Los Angeles, January 6, 1930.

$$\begin{array}{c|cc} u_1 & \eta_1 & \eta_2 \\ u_2 & \eta_1 & \eta_2 \, . \\ u_3 & \omega_1 & \omega_2 \end{array}$$

Moreover, these (and their integral combinations) are the only periods. For, let (P_1, P_2, P_3) be an arbitrary period. Then

$$u_1 + P_1 - \zeta(u_3 + P_3) = u_1 - \zeta(u_3),$$
$$u_2 + P_2 - \zeta(u_3 + P_3) = u_2 - \zeta(u_3).$$

Hence $P_1 = P_2$ and

$$\zeta(u_3 + P_3) - \zeta(u_3) = P_1.$$

In order that the function on the left-hand side of this identity admit no poles, it is necessary and sufficient that

$$P_3 = m_1\omega_1 + m_2\omega_2,$$

where m_1, m_2 are whole numbers. But then

$$P_1 = P_2 = m_1\eta_1 + m_2\eta_2.$$

Consider now an arbitrary linear combination of these functions. Such a function has the value

$$AF(u_1, u_2, u_3) + B\Phi(u_1, u_2, u_3) = (Au_1 + Bu_2) - (A + B)\zeta(u_3).$$

It is seen to depend on fewer than three linear combinations of u_1, u_2, u_3, for if we set

$$w_1 = Au_1 + Bu_2, \quad w_2 = u_3,$$

the function becomes

$$w_1 + (A + B)\zeta(w_2).$$

Hence the function $AF + B\Phi$ admits infinitely small periods. and the proof is complete.

HARVARD UNIVERSITY