ON THE TRANSFORMATION WHICH LEADS FROM THE BRIOSCHI QUINTIC TO A GENERAL PRINCIPAL QUINTIC*

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There are two well known methods of showing that any quintic equation (subject to certain restrictions which will not be gone into here) can be reduced to the important Brioschi normal form

$$w^5 - 10Zw^3 + 45Z^2w + k = 0, \tag{1}$$

with the aid of no irrationalities other than two square roots. Both methods employ a preliminary transformation to reduce the given quintic to the so-called principal form

$$Y^5 + 5aY^2 + 5bY + c = 0. \tag{2}$$

The coefficients of this transformation will involve one square root, in general, but no other irrationality. One method, that devised by Gordan † and later improved by Weber‡ and Dickson§, then sets up a transformation of the form $w = R(Y)$|| which leads from (2) to (1). This is, of course, the direct process, and the one which we should expect to follow, as a rule. However, in the present case, the required transformation is not at all simple, it is not easy to set up, and the constant term in the transformed equation is very difficult to determine explicitly.

Under the circumstances, it is by no means out of place to consider the second method, which, starting with equation(1), devises a transformation which leads from it to a principal quintic which can be identified with (2). It is then possible, by a known process, to set up a transformation leading from

* Presented to the Society, June 20, 1929.

† Mathematische Annalen, vol. 28 (1887), pp. 152–166.

‡ *Algebra*, 2d ed., vol. I, 1898, pp. 263–267.

§ *Modern Algebraic Theories*, 1926, pp. 214–218.

|| $R(Y)$ is a rational function of Y, the coefficients of which are rational functions of the coefficients of (2) and the square root of the discriminant of (2).

(2) back to (1), provided (2) does not have a double root. This second method has several distinct advantages. It employs a considerably simpler transformation than does the first method, namely

$$(3) \qquad Y = \frac{\lambda + \mu w}{Z^{-1}w^2 - 3},$$

λ and μ being parameters and k in (1) being put equal to $-Z^2$. The coefficients of the transformed equation (2) are then obtained from

$$\begin{aligned} Va &= 8\lambda^3 + \lambda^2\mu + 72\lambda\mu^2 Z + \mu^3 Z, \\ (4) \qquad Vb &= -\lambda^4 + 18\lambda^2\mu^2 Z + \lambda\mu^3 Z + 27\mu^4 Z^2, \\ Vc &= \lambda^5 - 10\lambda^3\mu^2 Z + 45\lambda\mu^4 Z^2 + \mu^5 Z^2, \end{aligned}$$

where V is written in place of $1728 - 1/Z$. Dickson shows how to identify the transformed equation with an arbitrary principal quintic by solving equations (4) for λ, μ, Z in terms of a, b, c.* The solution involves only one irrationality, again the square root of the discriminant of (2).

What this does is to tell us which particular equation (1) will transform into an arbitrary principal quintic, so we may say that the explicit form of both the original and the transformed equation is easy to determine. This is another advantage over the first method. Finally, while it is possible, as has been mentioned above, to set up, using (3), a transformation interchanging the role of our original and transformed equation, this need not be done explicitly. For it is possible that (3), which gives a root of any principal quintic in terms of a root of a certain Brioschi quintic, is a more useful relation than that given by the inverted transformation.

Until very recently, however, no direct, simple proof had been given showing that (3) led to a transformed equation with coefficients determined by (4). Gordan, who seems to be the first to obtain (3), or rather an equation essentially equivalent to it, † secured it merely as a corollary to a long discussion of other matters, chiefly icosahedral forms. A similar statement applies to Dickson's presentation. Kiepert‡ uses

* Ibid., pp. 244–247. This also covers the next reference to Dickson, below.

† Mathematische Annalen, vol. 13 (1878), pp. 375–404.

‡ Journal für Mathematik, vol. 87 (1879), pp. 114–133.

(3) and (4) in his work on the solution of the quintic by elliptic functions, but does not show how the transformed equation is set up. Heymann, in 1894, derived the transformed equation directly from the transformation, but his method is long and, in part, rather complicated.* A much improved treatment which may be based, to a certain extent, on that of Heymann was given by Perron, in 1927.† It seems to be the first simple, independent treatment of the matter.

Perron considers (1) with $Z=1$, but leaves k to be determined later. His transformation is

$$Y = \frac{\alpha}{w - 3^{1/2}} + \frac{\beta}{w + 3^{1/2}}, \tag{5}$$

which is essentially equivalent to (3). He sets up the transformed equation with the aid of values which he is able to obtain for expressions of the form

$$\sum \frac{1}{(w - 3^{1/2})^k (w + 3^{1/2})^l},$$

the summation being over the five roots of (1). Since his parameters α and β are not the λ, μ of (3), he does not obtain equations (4), but three somewhat similar equations.

The main purpose of this article is to give another treatment of the matter, one which is simple and direct and which leads to equations (4). These are, I believe, preferable to those of Perron, in that they can be solved somewhat more nicely for λ, μ, Z. We shall use (1) with $k=-Z^2$, and apply to this the transformation

$$v = w^2 Z^{-1} - 3. \tag{6}$$

The equation in v can be set up very easily by a device used in elementary work on equations; transpose the constant term of (1) to the right, factor w out of the left side, replace even powers of w, using

$$w^2 = Z(v + 3), \tag{7}$$

square both sides, replace w^2 and simplify. The result is

* Zeitschrift für Mathematik und Physik, vol. 39 (1894), pp. 162–182.

† *Algebra*, vol. 2, 1927, pp. 209–216.

$$Z(v+3)(v^2-4v+24)^2=1, \tag{8}$$

or

$$v^5-5v^4+40v^3+V=0,\ V=1728-1/Z. \tag{9}$$

The step just before (8) gives w rationally in terms of v,

$$w(v^2-4v+24)=1, \tag{10}$$

or using (8),

$$w=Z(v^3-v^2+12v+72). \tag{11}$$

We must now apply to (9) the transformation

$$Y=\frac{\lambda+\mu w}{v}, \tag{12}$$

where w must be replaced by its value in terms of v. We do not substitute (11) explicitly into (12), since (7) will be used as well. In setting up the transformed equation the sums of certain powers of the roots of (9) are needed; in the usual notation these are

$$s_2=-55,\ s_1=5,\ s_{-1}=s_{-2}=0,\ s_{-3}=\frac{-120}{V},\ s_{-4}=\frac{20}{V}. \tag{13}$$

If we now sum (12), using (11), for the roots of (9), we have, dropping at once the terms in s_{-1} and s_{-2}, $\sum Y=\mu Z(s_2-s_1+60)$, which is obviously zero. If we square (12), substitute from (7) and (11), and sum we find $\sum Y^2=2\lambda\mu Z(s_1-5)=0$. Hence the transformed equation has no terms in Y^4 and Y^3.

When (12) is cubed, w and w^2 replaced as before and w^3 by its value $Z^2(v^4+2v^3+9v^2+108v+216)$ from (7) and (11), it becomes

$$\begin{aligned} Y^3 = {} & \lambda^3v^{-3}+3\lambda^2\mu Z(1+72v^{-3})+3\lambda\mu^2Z(3v^{-3}) \\ & +\mu^3Z^2(v+2+216v^{-3}) \\ & + \text{terms in } v^{-1} \text{ and } v^{-2}. \end{aligned} \tag{14}$$

Now $\sum Y^3$ can be found very easily by using (13) and reducing the terms which do not involve s_{-3} to the denominator V, using, however, in the numerator the value $1728-1/Z$ in place of V. From the relation $\sum Y^3=-15a$ we are then led to the first equation (4).

Similarly, when (12) is raised to the fourth power, and w^4 replaced by its value from (7), we have

$$(15)\ Y^4 = \lambda^4 v^{-4} + 4\lambda^3\mu Z(12v^{-3} + 72v^{-4}) + 6\lambda^2\mu^2 Z(v^{-3} + 3v^{-4}) + 4\lambda\mu^3 Z^2(1 + 108v^{-3} + 216v^{-4}) + \mu^4 Z^2(6v^{-3} + 9v^{-4}) + \text{terms in } v^{-1} \text{ and } v^{-2}.$$

When this is summed and (13) used, every term but $20\lambda\mu^3Z^2$ has the denominator V. Change it as above, employ the relation $\sum Y^4 = -20b$, and the second equation of (4) is obtained.

It is of course possible to determine c in the same way from $\sum Y^5$, but a simpler method is available. For c is also the negative of the product of the roots of (2), or from (12),

$$(16)\qquad c = -\frac{\prod(\lambda + \mu w)}{\prod v}.$$

Now by (9) the denominator of (16) has the value $-V$. The numerator is, by definition, the resultant of the left side of (1) and $\lambda + \mu w$. By a familar theorem on resultants this is equal to the negative of the resultant of the two polynomials taken in the other order, which gives at once the value

$$(17)\qquad -\mu^5\left[\left(-\frac{\lambda}{\mu}\right)^5 - 10Z\left(-\frac{\lambda}{\mu}\right)^3 + 45Z^2\left(-\frac{\lambda}{\mu}\right) - Z^2\right].$$

The third equation of (4) follows at once from (16) and (17), and the transformed equation is completely determined.

This determination, together with Dickson's treatment of equations (4), gives a direct, self-contained proof of the theorem mentioned at the beginning of the paper. It remains simply to mention briefly the exceptional cases that may arise. For (3) to be valid, Z must not be zero, and $Z^{-1}w^2 - 3$ must not vanish for a root of (1). Both of these cases are covered if we require that (1) have no multiple root. A similar restriction will be made on (2), to insure the existence of a transformation leading from (2) back to (1). Finally, certain minor restrictions must be made in solving equations (4) for λ, μ, Z. Perron, in treating his system of equations similar to (4), mentions that the necessary restrictions characterize certain simple solvable quintics. A similar statement may be made for equations (4).

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