The Theory of Determinants, Matrices, and Invariants. By H. W. Turnbull. London and Glasgow, Blackie, 1928. 8 vo, vii+338 pp.

Mechanical Features. This volume is attractively bound in smooth dark green cloth over extra heavy boards. It is printed from new and artistic type on special quality glazed paper and represents a real achievement in scientific printing on account both of the elegant arrangement of the formulas on the page and of the perfection of typography. The concise German symbolical notation for determinants and algebraic invariants, used consistently, afforded to the printers an opportunity to manufacture an exceptional mathematical book.

The Book as a Text Book. Scope. The reader of the work will do well to be reminded of the full title, as otherwise the impression might be gained that extensive study of the theory of determinants is necessary before invariant theory can be undertaken. This is hardly true literally.

We think that the author of this admirable book was not concerned primarily to produce a text-book. He probably was more interested in a scientific question; that of creating a logically developed exposition of the portion of invariantive algebra which has its foundations largely in linearity and the linear properties of determinants and matrices. In carrying out this plan he was able to draw relevant material from a wide range of original sources, and has produced a work very likely to make a place for itself as a fundamental text. The book also is very stimulating reading for one who has read the original memoirs first.

In confining ourselves to linearity in invariant theory we necessarily omit practically all of reduction to fundamental systems, and syzygetic dependence. There is naturally, however, a chapter on Hilbert's Lemma on the basis of a system of polynomials, and the consequent proof of Gordan's theorem. A compensating advantage inherent in the plan is that most linear properties, such as, for example, the Laplace identities, are immediately extensible by generalization from two or three variables or sets to n variables, the generalization of the binary theory in those respects where generalization is feasible. The extension of all theories to the consequences of a literal number of variables is a characteristic feature of Professor Turnbull's work.

There are twenty-one chapters, with titles as follows: Matrices and determinants. Fundamental properties of the determinant. Linear properties (and) fundamental Laplace identities. Multiplication of matrices and determinants. Linear equation (and) corresponding matrices. Special types of determinant. Differentiation of a determinant. Binary forms. The general linear transformation. General properties of invariants. The first fundamental theorem. Multilinear forms. Symbolic methods of reduction. Seminvariants (and) algebraically complete systems. The Gordan-Hilbert finiteness theorem. Clebsch's theorem. Applications of Clebsch's theorem, canonical forms, etc. Invariant equations and Gram's theorem. Geometric interpretations. The general quadric. Miscellaneous recent developments.

Numerous and excellent lists of examples are interspersed and essential references are included.

The Aronhold-Clebsch symbolism is both so expressive and so concise and the author possesses such skill in all that pertains to elegance of notation that this book gives to casual reading the impression of being more elementary than it actually is. The author is persistent in giving the algorithms their simplest expression however. He uses the King's English happily and with simplicity and austerity.

Critical Evaluation. Allowing for the fact that a good book may, after the lapse of years, be superseded in whole or in part by another good book based upon the newer researches it remains true that a new work on invariantive algebra must justify its existence among the following competitors: Andoyer, *Leçons sur la Théorie des Formes*; Clebsch, *Theorie der Binären Formen*; Clebsch-Lindemann, *Vorlesungen ueber Geometrie*; Dickson, *Algebraic Invariants*; Elliott, *Algebra of Quantics*; Faà di Bruno, *Théorie des Formes Binaires*; Glenn, *Theory of Invariants*; Gordan, *Vorlesungen ueber Invariantentheorie*; Grace and Young, *Algebra of Invariants*; W. F. Meyer, *Allgemeine Formen und Invariantentheorie*; Salmon, *Lessons Introductory to Modern Higher Algebra*; Study, *Methoden zur Theorie der Ternären Formen*; Weitzenböck, *Invariantentheorie*; a book by Brioschi; and perhaps some others.

To be sure if we restrict our consideration to books in English, this book, due to its scope, would have few competitors if any.

The test of the newness of a new treatise is whether it offers a duplication of the master theorems presented in other books. If it does so it can have great value in the direction of popularization, scientific organization, pedagogy, etc., but not extensively as an original. We find in Turnbull's book a fairly novel layout of master theorems. This was possible because the subject, although only 125 years old, is vast and the number of books relatively small. Gordan planned another volume but did not compose it. Lindemann's enlarged edition of Clebsch's *Vorlesungen ueber Geometrie* was stopped by the war, and the second volume of W. F. Meyer's *Allgemeine Formen und Invariantentheorie* did not appear. As the author suggests in his preface, also, much of invariant theory in n variables is of comparatively recent origin.

In early chapters the elementary reducing identities of the binary symbolic theory are extended to the Laplace identities, the Sylvester identity and their generalizations. Parallel developments dealing with matrices, the simplex, tensors and dual transformations of the Sylvester identity (Chap. 4) lay the foundations for the theory of transformation and reduction in later chapters. Chapter 7 is an essay on Capelli operators, Cayley operators, jacobians and substitutional analysis. In Chapters 9 and 21 there are sections on orthogonality, linear transformation with absolute quadric, Hermite's theorem, and the affine group. One of the notable intervals in the book is Chapter 16 dealing with Clebsch's theorem on the expression of forms in terms of cogredient sets of compound variables, and the Gordan-Capelli series. Very remarkable is the recent Lasker-Wakeford theorem giving necessary and sufficient conditions that a given form be canonical. The complete concomitant system of the general quadric (in n variables x) and any number of linear forms, is developed in Chapter 20.

The author would rescue quantitative substitutional analysis from the position of aloofness which it occupies in the memoirs of Frobenius and Young. With this proposition the reviewer agrees, howbeit formalism and arithmetization are different questions in the general problem.

O. E. Glenn