they do not distinguish. The first of these expressions is most naturally interpreted as a proposition, p, which asserts, "p is false"; so that we have a proposition directly about itself, in connection with which a genuine vicious circle arises. But the authors do not consider this case; instead, they take the most natural interpretation of the second of the foregoing expressions, according to which it is to be rendered: "There is one and only one proposition which I am asserting, and it is false." Clearly, this is often true; but it cannot be true if I assert it, nor can it be false; and yet a vicious circle does not arise, since all that follows is that I never do assert it. Nevertheless, the situation here in question is just as objectionable as the occurrence of a vicious circle, since it leads to the conclusion that a logically possible state of affairs is in fact impossible.

C. H. LANGFORD

NÖRLUND ON FINITE DIFFERENCES

Leçons sur les Équations Linéaires aux Différences Finies. By N. E. Nörlund. Paris, Gauthier-Villars, 1929. vi+153 pp.

The theory of finite difference equations arose in investigations by Lagrange and Laplace; but it is only recently that the properties of the solutions of these equations have been developed with any considerable detail. The modern researches on this subject have been inaugurated by Poincaré and Pincherle. One owes to Poincaré a remarkable theorem on the manner in which the solutions of a linear homogeneous equation in finite differences behave for very large values of the variable. This theorem has been the point of departure of several investigations. In his preface the author says: "In recent years the theory of finite difference equations has been developed by a large number of authors, among whom may be mentioned G. D. Birkhoff, H. Galbrun, E. Hilb, E. Bortolotti, O. Perron, R. D. Carmichael, J. Horn, K. P. Williams, A. Guldberg and G. Wallenberg. [To this list, of course, should be added the name of Nörlund himself.] The subject is too vast for it to be possible to give here an exposition of all the results obtained. The aim of this book is to put in evidence the essential properties of the solutions of linear homogeneous equations, on the one hand by aid of their development in factorial series, on the other hand by aid of certain methods of successive approximations due to G. D. Birkhoff and R. D. Carmichael." Chapters I–IV are devoted to a single linear equation, different hypotheses relative to the coefficients being made in the different chapters. Chapters V–VI treat similar problems for a system of linear equations.

The first chapter is devoted to general properties of linear equations, such as the existence theorems which are readily proved, adjoint equations, equations with second member, and the reduction of the order of an equation by means of known solutions. A large part of this chapter is elementary and is devoted to the preliminaries of the general theory. There is, however, an existence theorem of considerable importance, based on hypotheses of a broad general character. In the fifth chapter one finds a treatment of precisely similar questions for a system of linear equations, each of the first order. The first and fifth chapters serve, therefore, to give the preliminaries of the theory, the one for a single equation, the other for a system of equations.

The second chapter is devoted to the solution of a single linear equation by means of factorial series. The necessary theory of factorial series is not developed in this book. It is supposed that the reader is already acquainted with the essential properties of these series. The definitions and results employed in the present exposition have been set forth by Nörlund himself in his book (Paris, 1926) entitled *Leçons sur les Séries d'Interpolation*, pp. 170–227. In many respects the theory developed in Chapter II is analogous to the classic theory of Fuchs for linear differential equations. Several of the artifices due to Fuchs and Frobenius in this latter theory have a similar application in the theory of difference equations. In some respects the existence theorems developed in this chapter are the most pleasing theorems in the whole field of the difference calculus.

The third chapter is devoted to an application of the transformation of Laplace to linear equations whose coefficients are polynomials. One of the tools required in the investigation is that of factorial series, so that this chapter has intimate contacts with the previous one.

On a first reading it is possible to omit Chapters II and III and pass immediately from Chapter I to Chapters IV, V and VI; and this can be done without destroying the continuity of thought. This fact may be useful to one who desires a knowledge of the main properties of the solutions of linear difference equations and who is unacquainted with the theory of factorial series.

In Chapter VI the solution of a linear equation by successive approximations is treated. The form of the method is that due to Carmichael and the detailed development follows closely his memoir of 1916 in the American Journal of Mathematics. The method is closely similar to and is an extension of the method already employed by the same author in the Transactions of this Society in 1911.

The final Chapter VI contains an exposition of the method of Birkhoff, principally as developed by him in 1911.

This book, though brief, contains the best-balanced introductory exposition of the theory of difference equations which has yet been prepared. The account is too short for extended details, but it is sufficiently comprehensive to give the reader an introduction to the various aspects of the theory of linear difference equations. A learner who is approaching this subject for the first time and who is equipped with the customary tools of modern analysis will find this book a convenient one to read early in his study of the subject. He might perhaps precede it by the reading of a portion of the still briefer book by Batchelder. He would do well to follow this present book by a study of the much more comprehensive treatise by Nörlund. With these three books before him, the student is well supplied with what is necessary for an introductory acquaintance with this modern subject of analysis. After a suitable study of these books he would naturally go then to the original memoirs.

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