ON A FUNCTION CONNECTED WITH $\phi(n)$

BY S. SIVASANKARANARAYANA PILLAI

Let $\phi(n)$ be the number of numbers not greater than and prime to n. Further, let

$$\begin{aligned} \phi_1(n) &= \phi(n), \\ \phi_2(n) &= \phi\{\phi_1(n)\}, \\ \phi_3(n) &= \phi\{\phi_2(n)\}, \\ &\cdots\cdots\cdots \\ \phi_{r+1}(n) &= \phi\{\phi_r(n)\}. \end{aligned}$$

It is obvious that at one stage

$$\phi_r(1) = 1.$$

Hence, with an integer n, there is associated another $R(n) = r$, such that r is the least integer for which

$$\phi_r(n) = 1.$$

This short note is about this function $R(n)$.*

THEOREM I.

$$R(n) \leqq \left[\frac{\log n}{\log 2}\right] + 1.$$

PROOF. When $n > 2$, $\phi(n)$ is always even. If x and y are even, and y contains at least one odd prime factor and x does not, then

$$\frac{\phi(x)}{x} > \frac{\phi(y)}{y}.$$

Hence, when n is even, $R(n)$ is maximum only if n is a power of 2.

If n is a power of 2,

* This problem was suggested by R. Vaidyanathaswami.

$$\frac{\phi(n)}{n}=\frac{1}{2},$$

and $\phi(n)$ is again a power of 2. Hence,

$$2^{R(n)}=n.$$

Therefore,

$$R(n)=\frac{\log n}{\log 2}.$$

But, by (1), $\phi(n)$ cannot be odd when $n>2$.

Hence, the maximum is attained when

$$n=p_1p_2\cdots p_s,$$

where $p_1, p_2, \cdots, p_s$ are primes of the form 2^l+1. Hence, if there are infinitely many primes of the form 2^l+1, then there are infinitely many n's for which

$$R(n)>\frac{\log n}{\log 2}+1-\epsilon$$

for every positive ϵ.

But even in this case,

$$R(n)\leqq\frac{\log n}{\log 2}+1.$$

But there are infinitely many values of n for each of which $R(n)=\log n/\log 2$, namely, $n=2^r$.

THEOREM II. *If n is such that $R(n)<R(n')$ for every $n'>n$, let us define n as H. Then*

$$H=2\cdot 3^r.$$

To prove this, we want two lemmas.

LEMMA 1. *If x is even,*

$$R(2x)=R(x)+1.$$

The proof is obvious.

LEMMA 2. *Whether x is odd or even, a multiple of 3, or not,*

$$R(3x)=R(x)+1.$$

provided that $x\geqq 2$.

Since $\phi(3)=2$, this easily follows from Lemma 1.

Now we are in a position to prove Theorem II. Put

$$2m = m_1, \qquad m_1 p = n_1,$$

where p is an odd prime greater than 3. As long as $\phi_{r-1}(m_1)$ is divisible by p,

$$\frac{\phi_r(m_1)}{m_1} = \frac{\phi_r(n_1)}{n_1}\,.$$

Let $\phi_{r-1}(m_1)$ be divisible by p, and $\phi_r(m_1)$ not. Then, obviously,

$$\phi_r(n_1) = \phi_r(m_1) \times p$$

and

$$\phi_{r+1}(n_1) = \phi_{r+1}(m_1) \times (p-1) = 2\phi_{r+1}(m_1)\cdot\frac{p-1}{2} = 2\cdot u \text{ (say)}.$$

Since $\phi_{r-1}(m_1)$ is divisible by the odd prime p which is >3, and $\phi_{r-1}(m_1)$ is even, $\phi_r(m_1)$ is even and is greater than 2. Hence, $\phi_{r+1}(m_1)$ is even. So, whether $(p-1)/2$ is odd or even, u is even. Therefore, by Lemma 1,

$$R(\phi_{r+1}(n_1)) = R(2u) = R(u) + 1\,.$$

But

$$\frac{\phi_{r+1}(n_1)}{\phi_r(n_1)} = \frac{\phi_{r+1}(m_1) \times (p-1)}{\phi_r(m_1) \times p}$$

$$= \frac{2\phi_{r+1}(m_1)}{\phi_r(m_1)} \times \frac{p-1}{2p};$$

and

$$\frac{p-1}{2p} \geqq \frac{5-1}{2\cdot 5} = \frac{4}{10} > \frac{1}{3}\,.$$

Hence, before the rate of decrease of $\phi_r(n_1)$ becomes 3 times that of $\phi_r(m_1)$, the value of $R(n_1)$ will be increased at least by one. This is the case at every stage where the rate of decrease is increased due to an odd prime in $\phi_r(n_1)$, which is not in $\phi_r(m_1)$. Hence, if

$$\frac{\phi_r(n_1)}{n_1} \leqq \frac{\phi_r(m_1)}{m_1} \times \frac{1}{3^l},$$

then

$$R(\phi_r(n_1)) \geqq R(\phi_r(m_1)) + l.$$

Hence we may write

$$\begin{aligned} R(n_1) &\geqq R(m_1) + \left[\frac{\log p}{\log 3}\right] + 1 \\ &= R(m_1) + k \text{ (say)} \\ &= R(3^k m_1), \end{aligned}$$

by Lemma 2. However, we have

$$3^k m_1 > n_1.$$

Therefore, n_1 cannot be an H.

Therefore, in order that n may be an H, n should be divisible by no odd prime greater than 3. Hence, if n is an H,

$$n = 2^s \cdot 3^t.$$

But

$$\phi_1(2^s \cdot 3^t) = 2^s \cdot 3^{t-1},$$

when $s \geqq 1$,

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and

$$\phi_t(2^s \cdot 3^t) = 2^s.$$

Therefore

$$R(2^s \cdot 3^t) = t + s.$$

Hence

$$R(2^s \cdot 3^t) = R(2 \cdot 3^{t+s-1}).$$

But

$$2 \cdot 3^{t+s-1} > 2^s \cdot 3^t.$$

Hence, unless $s < 2$, n is not an H. So, $n = 2 \cdot 3^t$ or 3^t. But

$$R(2 \cdot 3^t) = t + 1 = R(3^t).$$

Hence

$$n = 2 \cdot 3^t.$$

THEOREM III.

$$R(n) \geqq \left[\frac{\log n - \log 2}{\log 3}\right] + 1.$$

PROOF. By Theorem II, $R(n)$ is a minimum, when $n = 2 \cdot 3^t$, and then $R(n) = t + 1$.

But

$$t = \left[\frac{\log n - \log 2}{\log 3}\right].$$

Therefore

$$R(n) \geqq \left[\frac{\log n - \log 2}{\log 3}\right] + 1.$$

Hence

$$\left[\frac{\log n}{\log 2}\right] \geqq R(n) - 1 \geqq \left[\frac{\log \frac{n}{2}}{\log 3}\right].$$

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MULTIPLE POINTS OF ALGEBRAIC CURVES*

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1. *Introduction.* Limits to the number of multiple points of algebraic curves were first found by Cramer.† He found and tabulated the maximum numbers of multiple points of all possible orders for curves of orders up to and including eight. Plücker‡ obtained the general expression $(n-1)(n-2)/2$ for the maximum number of double points of an algebraic curve of order n.

Except for individual curves, the maximum number of multiple points of higher order than two for a curve of given order has not been found. A general expression for the maximum number of compound singularities or singularities of different orders is not practicable. When, however, the curve possesses only multiple points or sets of multiple points of the same order, serviceable limits for the maximum number of such singularities can be found.

The purpose of this paper is to determine the maximum number of distinct multiple points of given order and con-

* Presented to the Society, June 20, 1929.

† G. Cramer, *Introduction à l'analyse des lignes courbes algébriques*, Geneva, 1750, pp. 455–459.

‡ J. Plücker, *Theorie der Algebraischen Curven*, Bonn, 1839, p. 215.