THE DETERMINATION OF PLANE NETS CHARACTERIZED BY CERTAIN PROPERTIES OF THEIR LAPLACE TRANSFORMS*

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In a previous number of this Bulletin,† Professor J. O. Hassler discusses plane nets whose first and minus first Laplace transforms each degenerate into a straight line, and finds their canonical differential equations. The determination of these equations requires the solution of two partial differential equations of the first order in two dependent and two independent variables. Since two of Hassler's conditions are $H=K=0$, it follows from the well known theory of the equation of Laplace that the second of equations (1) below can be integrated by quadratures. It will be shown that the entire system (1) can be integrated by quadratures, and a fundamental set of integrals for the canonical system will be obtained.

In order to avoid unnecessary repetition of explanations and computations already contained in Hassler's paper, or in works to which he refers, we shall confine our attention entirely to the integration of the differential equations of the problem. These equations have the form

$$(1) \qquad \begin{cases} y_{11} = a^{(11)}y_1 + b^{(11)}y_2 + c^{(11)}y, \\ y_{12} = a^{(12)}y_1 + b^{(12)}y_2 + c^{(12)}y, \\ y_{22} = a^{(22)}y_1 + b^{(22)}y_2 + c^{(22)}y, \end{cases}$$

where $y_1=\partial y/\partial u$, $y_2=\partial y/\partial v$, etc., and where $a^{(ij)}$, $b^{(ij)}$, $c^{(ij)}$, $(i, j=1, 2)$, are functions of u and v, defined in the following manner:

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(2)\quad \begin{cases}
a_1^{(22)} + a^{(22)}b^{(11)} = 0, \quad b_2{}^{(11)} + a^{(22)}b^{(11)} = 0, \\
a^{(11)} = \dfrac{1}{3}(b^{(11)} - 1), \quad c^{(11)} = \dfrac{2}{9}(1 + b^{(11)} + [b^{(11)}]^2) \\
\qquad - \dfrac{1}{3}(b_1{}^{(11)} + b_2{}^{(11)}), \\
a^{(12)} = \dfrac{1}{3}(1 - a^{(22)}), \quad b^{(12)} = - a^{(11)}, \\
c^{(12)} = \dfrac{1}{9}(2a^{(22)}b^{(11)} + a^{(22)} + b^{(11)} - 1), \\
b^{(22)} = - a^{(12)}, \quad c^{(22)} = \dfrac{2}{9}(1 + a^{(22)} + [a^{(22)}]^2) \\
\qquad - \dfrac{1}{3}(a_1{}^{(22)} + a_2{}^{(22)}) .
\end{cases}
$$

Thus all the functions $a^{(ij)}$, $b^{(ij)}$, $c^{(ij)}$ are expressible in terms of the two $a^{(22)}$, $b^{(11)}$, and their derivatives, while these are defined by the first two equations. The relations defining $a^{(22)}$, $b^{(11)}$ show that

$$b^{(11)}du + a^{(22)}dv = d\psi(u,v),$$

and

$$a^{(22)} = \psi_2, \qquad b^{(11)} = \psi_1, \qquad a_1{}^{(22)} = b_2{}^{(11)} = \psi_{12}.$$

Hence we require the solution of

$$\psi_{12} + \psi_1\psi_2 = 0.$$

On integrating with respect to u and v, in turn, we find

$$e^{\psi}\psi_2 = A(v), \quad \psi = \log (\alpha(u) + \beta(v)),$$

where $\alpha(u)$, $\beta(v)$ are arbitrary functions of their respective arguments. The functions $a^{(22)}$, $b^{(11)}$ are therefore* given by

$$(3)\qquad a^{(22)} = \frac{\beta'}{\alpha + \beta}, \quad b^{(11)} = \frac{\alpha'}{\alpha + \beta},$$

the primes indicating differentiation.

* Hassler acknowledges his indebtedness to G. E. Raynor for these solutions, but does not indicate the method by which they were obtained.

The invariants of the second of equations (1) are

$$-H \equiv a_1^{(12)} - a^{(12)}b^{(12)} - c^{(12)} = -\tfrac{1}{3}(a_1^{(22)} + a^{(22)}b^{(11)}) = 0,$$

$$-K \equiv b_2^{(12)} - a^{(12)}b^{(12)} - c^{(12)} = -\tfrac{1}{3}(b_2^{(11)} + a^{(22)}b^{(11)}) = 0.$$

We now transform equations (1) by means of the substitution

$$y = e^{-V}Y,$$

where

$$V = -\int b^{(12)}du + a^{(12)}dv.$$

The system (1) becomes

$$(4)\qquad \begin{cases} Y_{11} + Y_1 - b^{(11)}(Y_1 + Y_2 + Y) = 0, \\ \qquad\qquad\qquad\qquad\qquad Y_{12} = 0, \\ Y_{22} + Y_2 - a^{(22)}(Y_1 + Y_2 + Y) = 0. \end{cases}$$

From the second of (4) we find

$$Y = G(u) + H(v).$$

Eliminate $Y_1 + Y_2 + Y$ between the first and third of equations (4), and take cognizance of (3); there results the equation

$$\frac{\alpha'}{G'' + G'} = \frac{\beta'}{H'' + H'}.$$

Since u and v are independent variables, we infer that

$$(5)\qquad G'' + G' = k\alpha\,, \quad H'' + H' = k\beta',$$

where k is a constant. On integrating these equations by the usual method, we find

$$G(u) = ke^{-u}\int e^u\alpha du + le^{-u} + l',$$

$$H(v) = ke^{-v}\int e^v\beta dv + me^{-v} + m',$$

where l, m, l', m' are additional constants. Hence the most general expression for Y is

$$Y = k\left[e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv\right] + le^{-u} + me^{-v} + l' + m'.$$

In order that the first and third of equations (4) may be satisfied by this value of Y, the constant $l'+m'$ must vanish. The equations (4) have therefore the following three linearly independent integrals:

$$Y^{(1)} = e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv,$$

$$Y^{(2)} = e^{-u}, \quad Y^{(3)} = e^{-v}.$$

If we substitute for $a^{(12)}$ and $b^{(12)}$ in the expression for V, and effect the resulting quadratures, we find

$$V = -\tfrac{1}{3}(u+v) + \tfrac{1}{3}\log(\alpha+\beta).$$

Consequently we have

$$y = \frac{e^{(u+v)/3}}{(\alpha+\beta)^{1/3}}Y.$$

Finally we have the integrals of the canonical differential equations of the most general plane net possessing the property that its first and minus first Laplace transforms each degenerate to a straight line in the form

$$y^{(1)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}\left[e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv\right],$$

$$y^{(2)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}e^{-u}, \quad y^{(3)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}e^{-v}.$$

Since α and β are arbitrary, it is possible to write these equations in a form entirely free from quadratures.

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