LANDAU ON NUMBER THEORY

Vorlesungen über Zahlentheorie. By Edmund Landau. Volume I: *Aus der elementaren und additiven Zahlentheorie.* xii+360 pp. Volume II: *Aus der analytischen und geometrischen Zahlentheorie.* vii+308 pp. Volume III: *Aus der algebraischen Zahlentheorie und über die Fermatsche Vermutung.* vii+341 pp. Leipzig, S. Hirzel, 1927.

These three excellently printed and arranged volumes form an addition of the highest importance to the literature of the theory of numbers. With them, the reader familiar with the basic elements of the theory of functions of a real and complex variable, can follow many of the astonishing recent advances in this fascinating field. His interest is enlisted at once and sustained by the accuracy, skill, and enthusiasm with which Landau marshals the analytic facts and simplifies as far as possible the inevitable mass of details.

Part I of Volume I gives a treatment of the elements of the theory of numbers up to the theory of quadratic residues and of the Pellian equation. This occupies a little over 50 pages. In Part II is first deduced the elementary inequality

$$a_1\xi/\log \xi < \pi(\xi) < a_2\xi/\log \xi$$

for the number of primes $\pi(\xi)$ not greater than ξ, after which follows the proof of the theorem of Brun (1919) that, if the number of "twin primes" p, $p+2$ is infinite, these are at least so infrequently distributed that the series of reciprocals

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots$$

converges, in contradistinction with the series of the reciprocals of all primes, which diverges. The reader is then introduced in the same part II to characters $\chi(a)$ mod k, and the associated Dirichlet series, by means of which is established Dirichlet's classical theorem that every arithmetical progression $l+ks(s=1, 2, \cdots)$ contains infinitely many primes if l and k are relatively prime.

All of this material in Parts I and II, which acquaints the reader with some of the most interesting ideas in number theory, occupies slightly less than 100 pages.

In Part III are obtained the classical theorems concerning the representation of an integer as the sum of two, three, or four squares, and in Part IV are developed the classical results concerning the class number of binary quadratic forms.

Part V presents that part of the recent extraordinary series of researches, *Some Problems of "Partitio Numerorum,"* Parts I–V (1920–1925), by Hardy and Littlewood, which lies in the direction of Goldbach's conjecture that every even integer >2 can be expressed as the sum of two odd primes. *Under the hypothesis* that not only Riemann's function $\zeta(s)$, but all of the similar functions defined by Dirichlet series, $L(s)$, have no zero for s with real part $>\Theta$ with Θ a definite constant $<\frac{3}{4}$, it is proved that every sufficiently great odd integer is representable as the sum of three odd primes, and that "almost all" even positive integers are the sum of two odd primes.

The sixth and last part of Volume I develops the part of the results of Hardy and Littlewood bearing on Waring's theorem (first proved by Hilbert, 1909), that any integer can be expressed as the sum of a fixed number g of nth powers, $g=g(n)$, in which they go a considerable distance toward determining the nature of the number theoretic function $G(n)\leqq g(n)$ representing the number G of powers sufficing for all large integers, and the function $G_1(n)\leqq G(n)$ representing the number G_1 sufficing for almost all large integers; for example, it is proved that

$$G(n) \leqq (n-2)2^{n-1}+5.$$

It is interesting that Landau's beautiful earlier result $G(3)\leqq 8$ (1909) escapes the analysis of Hardy and Littlewood who prove only $G(3)\leqq 9$.

At the end of this part is found Vinogradoff's brief proof (1922) of Waring's theorem, involving the use of Weyl's approximation theorems (1916), occupying in all less than 20 pages.

In Part VII, Volume 2, the principal result (Littlewood, 1924) is that the difference between the prime number function $\pi(x)$ and $\int_2^x (du/\log u)$ is of the order of

$$xe^{-\alpha(\log x \log\log x)^{1/2}}$$

at most. This result, as well as the analogous fact for any $\pi(x;k,1)$ is proved in a direct and simplified manner by Landau. Here the comparison with the earlier result of de la Vallée Poussin that this difference is of the order

$$xe^{-\alpha(\log x)^{1/2}}$$

at most, measures the progress made since the appearance of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen* (1909).

Part VIII of the same volume takes up thoroughly the number theoretic function $A(x)$, defined as the number of integer pairs (u, v) for which

$$u^2+v^2\leqq x,$$

that is, the number of points with integer cartesian coordinates within or on the boundary of the circle of radius $x^{1/2}$ and with center at the origin in the u, v plane. The fact that $A(x)-\pi x$ is of the order of $x^{1/2}$ at most is trivial. The major question is to determine as far as possible the order θ of this difference, the above trivial result showing that $\theta\leqq 1/2$. Here the facts established are $\theta\geqq 1/4$ (Hardy and Landau, 1915) and $\theta<1/3$ (Sierpinski, 1906; van der Corput, 1923). Similar results are deduced for other curves.

The first part of Volume 3 (Part XI) expounds the classical theory of algebraic numbers and ideals. Here the principal new development to be found is the proof of Thue's theorem (1908) that if $g(x, y)$ is an irreducible homogeneous polynomial of degree $n > 2$ with integral coefficients, the equation $g(x,y) = a$ for a given integral value of a has only a finite number of solutions. The main purpose of this part is to develop as much of the theory of ideals which is necessary to establish the known results concerning Fermat's famous conjecture that the equation

$$x^n + y^n = z^n$$

has no integral solution x, y, z with $xyz \neq 0$ for $n > 2$. In particular the proof of Kummer's theorem that the conjecture holds for $n = p$, a regular prime, is given in full. The concluding part deals with some recent results of Furtwängler (1912), Wieferich (1909), Mirimanoff (1910), and Vandiver (1914, 1919).

The mathematical world owes a great debt of gratitude to Professor Landau for rendering accessible so many of the recent splendid achievements in the theory of numbers.

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THREE BOOKS ON WAVE MECHANICS

Four Lectures on Wave Mechanics. By Erwin Shrödinger. London and Glasgow, Blackie and Son, 1928. viii+53 pp. Price 5 s.

Collected Papers on Wave Mechanics. By Erwin Schrödinger. Translated from the 2d German edition by J. F. Shearer and W. M. Deans. London and Glasgow, Blackie and Son, 1928. xiii+146 pp. Price 25 s.

Selected Papers on Wave Mechanics. By Louis de Broglie and Léon Brillouin. Translated by W. M. Deans. London and Glasgow, Blackie and Son, 1928. 151 pp. Price 15 s.

The wave mechanics proposed by de Broglie and Schrödinger has assumed during the short space of two years such a dominant role in the field of atomic physics that these three books in English by the authors of the new theory will be eagerly welcomed by physicists and mathematicians in England and America.

Schrödinger's *Four Lectures on Wave Mechanics* were delivered at the Royal Institution last March. In them are explained the ideas underlying the wave mechanics and some of the more important applications of these ideas. The exposition is somewhat popular in form, the mathematical details of computation being omitted from the text. The first lecture unfolds the relation between geometrical optics and classical dynamics as exhibited by the correspondence between Fermat's principle and the principle of Maupertuis (least action). It is this correspondence which suggests that classical methods are no more applicable to micro-mechanics than the ray methods of geometrical optics are to problems in diffraction and which leads in atomic problems to the substitution of a wave equation for the Hamilton-Jacobi equation of macro-mechanics. The success of a physical theory is measured by the agreement of its predictions with experi-