THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS IN A SPACE OF $(n+1)$ DIMENSIONS*

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1. *Introduction.* Among the various types of inverse problems of the calculus of variations are those of Darboux, Hamel, Hirsch, and Kürschak.† Darboux discussed the problem of the plane showing that for a given equation of the form $y''=\phi(x, y, y')$ there exist an infinity of functions $f(x, y, y')$ such that the integral $\int_{x_1}^{x_2} f(x, y, y')dx$ taken along one of the integral curves of the given equations furnishes a maximum or minimum. Hamel found the general type of integral whose minimizing arcs are straight lines. Of the last two Hirsch considers an equation of the type $F(x, y, y', y'', \cdots, y^{(n)})=0$ and Kürschak generalizes this by introducing n independent variables. In both of these cases it was found that a necessary and sufficient condition for a given equation of the type considered to give a solution of a problem in the calculus of variations is that it have its equation of variation self-adjoint. No such restriction was found in Darboux's problem; however, it is well known that every differential equation of the second order for plane curves may be transformed into one whose equation of variation is self-adjoint.

The inverse problem of the calculus of variations for three-dimensional space is treated in my thesis.‡ It is the

* Presented to the Society, San Francisco Section, June 2, 1928.

† Darboux, *Théorie des Surfaces*, vol. 3, §606.

A. Hirsch, *Ueber eine Charakteristische Eigenshaft der Differentialgleichungen der Variationsrechnung*, Mathematische Annalen, vol. 49, p. 49.

J. Kürschak, *Ueber die Transformation der partiellen Differentialgleichungen der Variationsrechnung*, Mathematische Annalen, vol. 56, p. 155.

G. Hamel, *Geometrieen, in denen die Geraden die Kurzesten sind*, Mathematische Annalen, vol. 59, p. 255.

‡ *Inverse problem of the calculus of variations in higher space*, written under the direction of G. A. Bliss, University of Chicago, 1926. Published in the Transactions of this Society, vol. 30 (1928), pp. 710–736.

purpose of this paper to discuss the corresponding problem for a space of $(n+1)$ dimensions.

2. *Fundamental Properties of given Differential Equations.* Let us consider a system of n differential equations of the form

$$H_j(x, y_i, y_i', y_i'') = 0, \quad (i, j = 1, \cdots, n), \tag{1}$$

whose solutions are

$$y_i = y_i(x);$$

these have the derivatives y_i', y_i'' with respect to x. Under the hypothesis that the equations of variation of the given equations (1) form a self-adjoint system,* a function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ can be determined such that the given equations are the differential equations for the solutions of the problem of minimizing the integral

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx. \tag{2}$$

The self-adjoint conditions which are needed here are summarized in the following theorem which is fully treated in the reference cited below.†

THEOREM. *Necessary and sufficient conditions that the system of differential expressions*

$$J_i(u) \equiv A_{ik}(x)u_k + B_{ik}(x)u_k' + C_{ik}(x)u_k'', \quad (i, k = 1, 2, \cdots, n),$$

shall be self-adjoint are

$$C_{ik} \equiv C_{ki},$$
$$B_{ik} + B_{ki} \equiv 2C_{ik}',$$
$$A_{ik} \equiv A_{ki} - B_{ki}' + C_{ki}''.$$

In the above and following expressions, the notation of

* This condition is also necessary; see Theorem II of my thesis, loc. cit., or J. Hadamard, *Leçons sur le Calcul des Variations*, p. 156.

† See my thesis, loc. cit.

tensor analysis is used, that is, whenever two subscripts are alike in two factors of a term, say of the form $A_{ik}u_k$, then the expression represents a sum with respect to the repeated index.

The equations of variation of the system (1) are

$$H_{iy_j''}u_j'' + H_{iy_j'}u_j' + H_{iy_j}u_j = 0. \tag{3}$$

For this system the self-adjoint conditions of the above theorem give respectively the following relations:

$$\begin{aligned} H_{iy_j''} &= H_{jy_i''}\,, \\ H_{jy_i'} + H_{iy_j'} &= 2(H_{iy_j''})', \\ H_{jy_i} &= H_{iy_j} - (H_{iy_j'})' + (H_{iy_j''})'', \end{aligned} \tag{4}$$

which must be identities in x, y_i', y_i', y_i'''.

The second set of relations (4) assert that each of the functions $H_i(i=1, \cdots, n)$ is linear in y_k'' $(k=1, \cdots, n)$, since terms in y_k''' do not occur in the first members. Therefore, the given functions (1) may be written in the form

$$\begin{aligned} H_i = M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\ + P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j''. \end{aligned} \tag{5}$$

In this notation the first of relations (4) becomes

$$P_{ij} = P_{ji}. \tag{6}$$

From the last two of relations (4) we have

$$H_{jy_i} - H_{iy_j} = [(H_{iy_j''})' - H_{iy_j'}]' = \tfrac{1}{2}(H_{jy_i'} - H_{iy_j'})'. \tag{7}$$

Since the coefficient of y_k''' in the expansion of the second member of this equation must vanish we have the following conditions:

$$H_{jy_i'y_k''} = H_{iy_j'y_k''}. \tag{8}$$

In the notation of (5) these relations become

$$P_{jky_i'} = P_{iky_j'}. \tag{9}$$

From these conditions and (6) it follows that the expression

$P_{jky_i'}$ remains unchanged under all permutations of the indices i, j, k.

From (4_2) with the aid of (5) and (9), we obtain

$$
(10)\qquad
\begin{aligned}
M_{iy_j'} + M_{jy_i'} &= 2P_{ij}' - P_{iky_j'}y_k'' - P_{jky_i'}y_k'' \\
&= 2(P_{ijx} + P_{ijy_k}y_k' + P_{ijy_k'}y_k'') \\
&\quad - P_{jky_j'}y_k'' - P_{jky_i'}y_k'' \\
&= 2(P_{ijx} + P_{ijy_k}y_k').
\end{aligned}
$$

By applying relations (7) to the expressions for H_i given in (5) and with the use of (9), we get

$$
(11)\quad M_{jy_i} - M_{iy_j} = (P_{iky_j} - P_{jky_i})y_k'' + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})'.
$$

If these equations are to be identities in the variables involved, we must have

$$
(11_a)\qquad
\left\{
\begin{aligned}
P_{iky_j} - P_{jky_i} &= \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_{y_k'}, \\
M_{jy_i} - M_{iy_j} &= \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_x \\
&\quad + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_{y_k}y_k'.
\end{aligned}
\right.
$$

The first of these systems may be obtained from (10) with the use of (9) but the second is an independent system of relations which must be satisfied.

The above results may be summarized as follows:

THEOREM 1. *If a system of differential equations of the second order*

$$
H_j(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0, \qquad (j = 1, \cdots, n),
$$

is to have equations of variation which are self-adjoint along every curve $y_i = y_i(x)$, then it must have the form

$$
\begin{aligned}
H_i = M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\
+ P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j'',
\end{aligned}
$$

$(i, j = 1, \cdots, n)$ *where the functions M_i and P_{ij} satisfy the conditions*

$$
(12) \qquad \begin{cases} P_{ij} = P_{ji}\,; \quad P_{jky_i'} = P_{iky_j'}, \\ M_{jy_i'} + M_{iy_j'} = 2(P_{ijx} + P_{ijy_k}y_k'), \\ M_{jy_i} - M_{iy_j} = \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_x \\ \qquad\qquad + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_{y_k}y_k', \end{cases}
$$

identically in x, y_i, y_i'.

3. *Determination of the Integrand f of the Integral I.* For a system of equations $H_j = 0$ which has the form indicated in the above theorem with coefficients satisfying (12), it is possible to determine the integrand f of the integral I in (2) to an arbitrary function of $x, y_1, \cdots, y_n$ such that $H_j = 0$ are the equations of its minimizing curves.

A function $g(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ will first be determined satisfying the relations

$$
(13) \qquad g_{y_i'y_j'} = P_{ij}, \qquad (i, j = 1, \cdots, n).
$$

The relations (12_2) are necessary and sufficient conditions for the integrability of system (13). The function g will be determined by the integral

$$
g = \int_{y_{10}, \cdots, y_{n0}}^{y_1', \cdots, y_n'} L_1 dy_1' + L_2 dy_2' + \cdots + L_n dy_n',
$$

where

$$
L_1 = \int_{y'_{10}, \cdots, y_n{}'_0}^{y_1', \cdots, y_n'} P_{11}dy_1' + P_{12}dy_2' + \cdots + P_{1n}dy_n',
$$

$$
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$$
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$$

$$
L_n = \int_{y'_{10}, \cdots, y'_{n0}}^{y'_1, \cdots, y_n'} P_{n1}dy_1' + P_{n2}dy_2' + \cdots + P_{nn}dy_n'.
$$

If g is a particular solution of the system (13), the most general solution is given by the formula

$$
(14) \quad f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x, y_1, \cdots, y_n) \\ + b_k(x, y_1, \cdots, y_n)y_k', \quad (k = 1, \cdots, n),
$$

where $a, b_1, \cdots, b_n$ are arbitrary functions of $x, y_1, \cdots, y_n$.

If the given equations are identical with the Euler-Lagrange equations for the integral (2) then the equations

$$\frac{d}{dx}(f_{y_i'}) - f_{y_i} = M_i + P_{ij}y_j'' \tag{15}$$

must be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n''$. The value of f given in (14) must now satisfy the system (15); thus we have

$$\frac{d}{dx}(g_{y_i'} + b_i) - (g_{y_i} + a_{y_i} + b_{ky_i}y_k') = M_i + P_{ik}y_k'',$$

which readily reduce to

$$(b_{ix} - a_{y_i}) + (b_{iy_k} - b_{ky_i})y_k' = M_i + g_{y_i} - g_{y_i'x} - g_{y_i'y_k}y_k'. \tag{16}$$

If these equations are to be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ the expressions on the right must be independent of y_i' and linear in y_k' when $k \neq i$. Upon differentiating the second members of (16) with respect to y_i' one obtains

$$M_{iy_i'} - g_{y_i'y_i'x} - g_{y_i'y_i'y_k}y_k',$$

which by conditions (12_3), for $i=j$, are identically zero. Differentiation with respect to y_j' $(j \neq i)$, gives

$$b_{iy_j} - b_{jy_i} = M_{iy_j'} + g_{y_iy_j'} - g_{y_i'y_j} - g_{y_i'y_j'x} - g_{y_i'y_j'y_k}y_k'.$$

With the aid of (13) and (12_3) this system reduces to

$$b_{iy_j} - b_{jy_i} = \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'}) + g_{y_iy_j'} - g_{y_i'y_j}. \tag{17}$$

If in these equations we again take the partial derivatives, with respect to y_j', the functions on the right yield the following expressions:

$$\tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_{y_j'} + P_{jjy_i} - P_{ijy_j}. \tag{18}$$

By setting $k=j$ in the first system of (11_a) we see that the expressions (18) are each identically zero. Thus we have shown that the second members of (16) are, as indicated by the expressions on the left, independent of y_i and linear in y_k for $k\neq i$.

Upon substituting the values of the first members of (17) in (16) we see that the system (16) which are partial differential equations for a, $b_1, \cdots, b_n$ may be replaced by the following:

$$
(19)\qquad
\begin{aligned}
b_{jy_i} - b_{iy_j} &= g_{y_i'y_j} - g_{y_iy_j'} - \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'}),\\
b_{jx} - a_{y_j} &= M_j + g_{y_j} - g_{y_j'x} - g_{y_jy_k'}y_k'\\
&\quad - \tfrac{1}{2}(M_{jy_k'} - M_{ky_j'})y_k'.
\end{aligned}
$$

In (19_1) there are $n(n-1)/2$ equations and in (19_2) n equations in the $n+1$ variables x, $y_1, \cdots, y_n$. The determinant of the matrix of the functional expressions on the right in (19_1) is skew symmetric. Solutions for the system (19) exist if the totality of $n(n+1)/2$ equations are compatible.

We shall now prove the following theorem.

THEOREM 2. *A necessary and sufficient condition that there exist a solution for a system of differential equations of the form*

$$
(20)\qquad b_{jy_i} - b_{iy_j} = \phi_{ij},\quad b_{jx} - a_{y_j} = \theta_j,
$$

where a, ϕ_{ij}, b_j, θ_j, $(i, j=1, 2, \cdots, n)$, *are functions of* x, $y_1, \cdots, y_n$ *and* $\phi_{ij} = -\phi_{ji}$, *is that*

$$
(21)\qquad \phi_{ijx} - \theta_{jy_i} + \theta_{iy_j} \equiv 0
$$

identically in x, $y_1, \cdots, y_n$ *for every pair of values of* i *and* j.

That the condition is necessary is evident. In order to prove that it is also sufficient consider the given equations (20) for fixed values of i and j, namely, $i=p$, $j=q$. For these values of i and j system (20) gives

$$
(22)\quad b_{qy_p} - b_{py_q} = \phi_{pq},\quad a_{y_q} - b_{qx} = -\theta_q,\quad b_{px} - a_{y_p} = \theta_p.
$$

Now since

$$
\phi_{pqx} - \theta_{qy_p} + \theta_{py_q} \equiv 0,
$$

identically in x, y_p, y_q, there exists a solution, say a,

b_p, b_q, of equations (22)*. Now let $j=r$, where r is any other value of j than q. The substitution of the functions a, b_p, b_r in the equations for $i=p$, $j=r$ gives

$$(23)\quad b_{ry_p} - b_{py_r} = \phi_{pr}, \quad a_{y_r} - b_{rx} = -\theta_r, \quad b_{px} - a_{y_p} = \theta_p.$$

Since the last of these equations is identical with the last equation of (22), it is satisfied by the functions a, b_p. Solving for the derivatives of b_r in the first two equations, we obtain

$$b_{ry_p} = \phi_{pr} + b_{py_r}, \quad b_{rx} = \theta_r + a_{y_r},$$

which are compatible provided that

$$\phi_{prx} + b_{py_rx} = \theta_{ry_p} + a_{y_ry_p}.$$

By employing the last of equations (22) this equation becomes

$$\phi_{prx} - \theta_{ry_p} + \theta_{py_r} = 0,$$

which is identically zero by hypothesis. A similar condition prevails for $i=s$, $j=q$, where s is any other value of i than p. It follows that the remaining $(n+3)(n-2)/2$ equations are compatible for a solution of the given equations for $i=p$, $j=q$. Hence, there exists a solution of the given system.

System (20) is identical with (19) if ϕ_{ij} and θ_j represent the second members of (19) respectively. Relations (21) applied to the second members of (19) give the following system:

$$(24)\quad 0 = (M_{iy_j} - M_{jy_i}) - \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_x + \tfrac{1}{2}(M_{jy_k'} - M_{ky_j'})_{y_i}y_k' + \tfrac{1}{2}(M_{ky_i'} - M_{iy_k'})_{y_j}y_k'.$$

These equations are identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$, provided the following system is identically zero:

$$(25)\quad (M_{jy_k'} - M_{ky_j'})_{y_i}y_k' + (M_{ky_i'} - M_{iy_k'})_{y_j}y_k' - (M_{jy_i'} - M_{iy_j'})_{y_k}y_k' \equiv 0,$$

* See H. Weber, *Die Partiellen Differentialgleichungen*, vol. 1, pp. 221, etc.

which is obtained by replacing the first two expressions in parenthesis of (24) by their value obtained from the last system of (12).

It is readily verified that those terms of (25) for which any two of the indices i, j, k are equal vanish identically. It remains to show that the terms for which $k \neq i \neq j$ will also vanish. For this purpose consider the last of the self-adjoint relations (12). Let this system be written successively for $j=p$, $i=r$; $j=q$, $i=p$; $j=r$, $i=q$. If these three equations are differentiated with respect to y_q', y_r', y_p' respectively and the results added, we obtain equations (25) except for the factor y_k'. Hence, equations (25) are identically zero in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$.

Since the conditions of Theorem 2 are satisfied there exists a solution of the system (19). Let a, b_i, b_j be a particular solution of system (19); then the most general solution will be

$$
\begin{aligned}
&a(x,y_1,\cdots,y_n)+u(x,y_1,\cdots,y_n),\\
(26)\qquad &b_i(x,y_1,\cdots,y_n)+v_i(x,y_1,\cdots,y_n),\\
&b_j(x,y_1,\cdots,y_n)+v_j(x,y_1,\cdots,y_n),\qquad (i<j),
\end{aligned}
$$

where the functions u, v_i, v_j satisfy the relations

$$
(27)\quad v_{jy_i}-v_{iy_j}=0,\quad u_{y_j}-v_{jx}=0,\quad v_{ix}-u_{y_i}=0,\quad (i>j).
$$

The general solution of the system (19) may now be written in the form

$$
\begin{aligned}
(28)\qquad &a(x,y_1,\cdots,y_n)+u(x,y_1,\cdots,y_n),\\
&b_k(x,y_1,\cdots,y_n)+v_k(x,y_1,\cdots,y_n),\\
&\qquad\qquad (k=1,\cdots,n).
\end{aligned}
$$

Relations (27) are necessary and sufficient conditions that the expression

$$
u+v_ky_k',\qquad (k=1,\cdots,n),
$$

should be the total derivative of an arbitrary function

$t(x, y_1, \cdots, y_n)$. In view of (28), the integrand f of (14) takes the form

$$(29)\ f = g(x,y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x,y_1, \cdots, y_n) + b_k(x,y_1, \cdots, y_n)y_k' + \frac{d}{dx}t(x,y_1, \cdots, y_n),$$

where a, $b_1, \cdots, b_n$ are solutions of the system (19) and t is an arbitrary function of x, $y_1, \cdots, y_n$.

THEOREM 3. *If a system of differential equations of the form*

$$H_i(x,y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0, \quad (i = 1, \cdots, n),$$

has equations of variation

$$H_{iy_j''}u_j'' + H_{iy_j'}u_j' + H_{iy_j}u_j = 0,$$

which form a self-adjoint system along every curve $y_i = y_i(x)$, *then there exists an integral of the form*

$$I = \int_{x_1}^{x_2} f(x,y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

having the equations $H_i(x, y_j, y_j', y_j'') = 0$ *as its Euler equations. The most general such integral has an integrand*

$$f = g(x,y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x,y_1, \cdots, y_n) + b_k(x,y_1, \cdots, y_n)y_k' + \frac{d}{dx}t(x,y_1, \cdots, y_n)$$

where g *is a particular solution of* (13), a, $b_1, \cdots, b_n$ *are a particular solution of system* (19), *and* t *is an arbitrary function of* x, $y_1, \cdots, y_n$.

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