ON THE ATTRACTION OF SPHERES IN ELLIPTIC SPACE*

BY JAMES PIERPONT

1. *Introduction*. C. Neumann, Killing, and Liebmann have treated the motion of a material particle about a center of attraction in elliptic (hyperbolic) space. The question arises do these results hold when the center of attraction is replaced by a spherical mass.

Let the sphere be placed at the origin of coordinates O, let the polar coordinates of an element of volume at P of the sphere be ρ, ϕ, θ, where ϕ, θ are latitude and longitude. The element of volume is then

$$dv = \sin^2 \rho \cos \phi d\theta d\rho d\phi,$$

where for simplicity we take the space constant $R=1$. We will suppose the elementary mass A attracted by the sphere is on the z axis. Let $OA=\alpha$, $AP=\epsilon$, in elliptic measure. The force of attraction we will take to be

$$F = \frac{cdv}{\sin^2 \epsilon}, \qquad c \text{ a constant.}$$

If ψ is the angle AP makes with the z axis, the work done by the force F for a small displacement of A of extent $\delta\alpha$ along the z axis is

$$\delta W = F \cos \psi dv \cdot \delta\alpha.$$

It will be convenient to set

$$a = \sin \alpha, \quad r = \sin \rho, \quad e = \sin \epsilon, \quad p = \sin \phi,$$

$$a' = \cos \alpha, \quad r' = \cos \rho, \quad e' = \cos \epsilon, \quad p' = \cos \phi.$$

We have then

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$$\cos\psi = \frac{(r' - a'e')}{ae},$$

$$e = (\alpha p^2 + \beta p + \gamma)^{1/2},$$

where

$$\alpha = -a^2r^2, \qquad \beta = -2aa'rr', \qquad \gamma = 1 - a'^2r'^2.$$

Then

$$\delta W = 2\pi c(aG - a'H)\delta\alpha,$$

where

$$G = \int_0^\rho r^2r'Ld\rho, \qquad L = \int_{-\pi/2}^{\pi/2} p'd\phi/e^3,$$

$$H = \int_0^\rho r^3Md\rho, \qquad M = \int_{-\pi/2}^{\pi/2} pp'd\phi/e^3.$$

Now

$$L = \left[\frac{4\alpha p + 2\beta}{\Delta(\alpha p^2 + \beta p + \gamma)^{1/2}}\right]_{-1}^{1}, \qquad \Delta = 4\alpha\gamma - \beta^2 = -4a^2r^2,$$

$$\alpha + \beta + \gamma = 1 - (a'r' + ar)^2,$$

$$(\alpha + \beta + \gamma)^{1/2} = \sin(\alpha - \rho),$$

$$(\alpha - \beta + \gamma)^{1/2} = \sin(\alpha + \rho),$$

$$4\alpha + 2\beta = -4ar\cos(\alpha - \rho),$$

$$4\alpha - 2\beta = 4ar\cos(\alpha + \rho).$$

Hence

$$L = \frac{2}{ar}\,\frac{\sin 2\rho}{\cos 2\rho - \cos 2\alpha}, \quad G = \frac{1}{a}\int_0^\rho \frac{\sin^2 2\rho d\rho}{\cos 2\rho - \cos 2\alpha}\,.$$

Turning to M, we have

$$M = -\left[\frac{4\gamma + 2\beta p}{\Delta(\alpha p^2 + \beta p + \gamma)^{1/2}}\right]_{-1}^{1},$$

$$4\gamma - 2\beta = 4(1 - a'r'\cos(\alpha + \rho)),$$

$$4\gamma + 2\beta = 4(1 - a'r'\cos(\alpha - \rho)).$$

Hence

$$M = \frac{M'}{a^2 r^2} + \frac{a'r'}{a^2 r^2} M'',$$

where

$$M' = \frac{-4 \sin \rho \cos \alpha}{\cos 2\alpha - \cos 2\rho}, \quad M'' = \frac{-2 \sin 2\rho}{\cos 2\rho - \cos 2\alpha}.$$

Hence

$$M = \frac{2a'}{a^2} \cdot \frac{r}{a^2 - r^2}.$$

Thus

$$G = \frac{2}{a} \int_{-1}^{1} \frac{r^2(1 - r^2)^{1/2}}{a^2 - r^2} dr,$$

$$H = \frac{2a'}{a^2} \int_{-1}^{1} \frac{r^4 dr}{T},$$

where $T = (a^2 - r^2)(1 - r^2)^{1/2}$. Then

$$aG - a'H' = 2 \int_0^r \frac{r^2(1 - r^2) dr}{T} - \frac{2}{a^2} \int_0^r \frac{r^2 dr}{T}$$

$$= \frac{2}{a^2} \int_0^r \frac{r^2 dr}{(1 - r^2)^{1/2}} = \frac{1}{2a^2}(2\rho - \sin 2\rho).$$

Now the volume of a sphere of radius ρ in elliptic measure is $V = \pi(2\rho - \sin 2\rho)$. As $m = cV$ is the mass of the sphere, this gives

$$\delta W = \frac{m}{a^2} \cdot \delta\alpha.$$

The attraction of the sphere on the point A is thus m/a^2, which is the same as if the mass of the sphere were concentrated at its center.

2. *Another Method.* Let us define the potential of an element of mass dm at a distance ϵ by $dU = c \operatorname{ctn} \epsilon \cdot dv$, c constant, and for the whole body supposed homogeneous, by

$$U = c \int \operatorname{ctn} \epsilon \cdot dv.$$

Then α being as in §1,

$$\frac{\partial U}{\partial \alpha} = c \int dv \cdot \frac{\partial}{\partial \alpha} \operatorname{ctn} \epsilon = c \int dv \cdot \frac{\partial \operatorname{ctn} \epsilon}{\partial \epsilon} \cdot \frac{\partial \epsilon}{\partial \alpha} \cdot$$

Let A' be the point on the z axis at a distance $\alpha + \delta\alpha$ from O. Let B lie on PA' at a distance ϵ from P. Then in the small triangle ABA' we have

$$\cos \psi = \frac{\sin BA'}{\sin AA'}, \qquad \text{or} \qquad \cos \psi = \frac{\partial \epsilon}{\partial \alpha} \cdot$$

Thus

$$-\frac{\partial U}{\partial \alpha} = c \int \frac{\cos \psi}{\sin^2 \epsilon} \cdot dv = \int F \cos \psi \cdot dv = \int F_z \, dv,$$

where F_z is the component of F along the z axis.

This result gives us another method of finding the attraction of a homogeneous sphere on a point at A. We have

$$\begin{aligned} U &= c \int \operatorname{ctn} \epsilon \, dv = c \int \frac{e'}{e} r^2 p' d\theta d\rho d\phi \\ &= 2\pi c(a' G_1 + a H_1), \end{aligned}$$

where

$$G_1 = \int_0^\rho r^2 r' L_1 d\rho, \qquad L_1 = \int_{-1}^{1} \frac{dp}{Q}, \; Q = (\alpha p^2 + \beta p + \gamma)^{1/2},$$

$$H_1 = \int_0^\rho r^3 M_1 d\rho, \qquad M_1 = \int_{-1}^{1} \frac{p dp}{Q} \cdot$$

Now

$$\begin{aligned} L_1 &= \frac{1}{(-\alpha)^{1/2}} \left\{ \sin^{-1} \frac{\beta - 2\alpha}{(-\Delta)^{1/2}} - \sin^{-1} \frac{\beta + 2\alpha}{(-\Delta)^{1/2}} \right\}, \\ \beta - 2\alpha &= -2ra(a'r' - ar) = -2ar \cos(\alpha + \rho), \\ \beta + 2\alpha &= -2ar \cos(\alpha - \rho). \end{aligned}$$

Hence

$$L_1 = \frac{1}{ar} \left\{ \sin^{-1}(\cos(\alpha + \rho)) - \sin^{-1}(\cos(\alpha - \rho)) \right\} = \frac{2\rho}{ar} \cdot$$

Similarly

$$M_1 = \left[\frac{(\alpha p^2 + \beta p + \gamma)^{1/2}}{\alpha}\right]_{-1}^{1} - \frac{\beta L_1}{2\alpha}$$

$$= \frac{1}{\alpha}\left\{(\alpha+\beta+\gamma)^{1/2} - (\alpha-\beta+\gamma)^{1/2}\right\} - \frac{\beta L_1}{2\alpha}$$

$$= \frac{1}{a^2r^2}\left\{\sin(\alpha+\rho) - \sin(\alpha-\rho)\right\} - \frac{2a'r'\rho}{a^2r^2} = \frac{2a'}{a^2r^2}(r - r'\rho).$$

Hence

$$G_1 = \frac{1}{4a}(\sin 2\rho - 2\rho\cos 2\rho),$$

$$H_1 = \frac{2a'}{a^2}(\tfrac{1}{2}\rho - \tfrac{1}{4}\sin 2\rho - \tfrac{1}{8}\sin 2\rho + \tfrac{1}{4}\rho\cos 2\rho).$$

Thus

$$U = \frac{a'}{2a}(2\rho - \sin 2\rho) = \frac{a'm}{a} = m \operatorname{ctn} a;$$

that is, the potential of a sphere at A is the same as if the whole mass were concentrated at its center.

3. *Attraction at a Point Within.* We consider now the potential W of a homogeneous spherical shell at a point within the cavity. This may be treated briefly. Proceeding as in §2, we have

$$W = 2\pi c(a'G_2 + aH_2),$$

where

$$G_2 = \int r^2r'L_2d\rho, \qquad H_2 = \int r^3M_2d\rho.$$

Here L_2, M_2 are the same as L_1, M_1 in §2 except that now we must take $\rho > \alpha$. Hence

$$L_2 = \frac{2\alpha}{ar} = -ar, \qquad M_2 = \frac{2r'}{ar^2} + a'r'.$$

These give

$$G_2 = -a\int_{\rho_1}^{\rho_2} r^3 dr, \qquad H_2 = \frac{2}{a}\int_{\rho_1}^{\rho_2} r dr + a'\int_{\rho_1}^{\rho_2} r^3 dr.$$

Hence

$$W = 4\pi c\int_{\rho_1}^{\rho_2} r dr$$

which is a constant. Thus the attraction on a point within the cavity is null.

4. *Conclusion.* We have supposed the sphere to be of uniform density. It is obvious from the above that we get the same result if the sphere is made up of concentric layers each of constant density, or if the density is a function of ρ. Finally we remark that the analysis employed may be extended readily to hyperbolic space.

In connection with the present note we wish to call attention to a valuable paper* contributed by F. Cajori to the memorial volume "Sir Isaac Newton 1727–1927." It is almost universally held that Newton's delay was due to the fact that the earth's radius was not correctly known at that epoch. Cajori argues conclusively, so we believe, that the real reason was much deeper. It was in fact Newton's inability at that time to show that the attraction of a sphere on an external particle is the same as if the mass of the sphere were concentrated at its center.

YALE UNIVERSITY

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* *Newton's twenty year's delay in announcing the law of gravitation.*