THE FEBRUARY MEETING IN NEW YORK

The two hundred sixty-seventh regular meeting of the Society was held at Columbia University, on Saturday, February 23, 1929, extending through the usual morning and afternoon sessions. The attendance included the following fifty-nine members.

R. L. Anderson, R. G. Archibald, H. E. Arnold, Benton, G. A. Campbell, J. A. Clark, Cramlet, Demos, Doermann, J. E. Donahue, Dresden, Engstrom, Fite, D. A. Flanders, R. M. Foster, Gehman, Gill, Glenn, Gourin, Gronwall, Hazeltine, L. S. Hill, Himwich, E. H. Johnson, M. I. Johnson, Kasner, Kenny, Kline, Koopman, Mark Kormes, Kramer, Marden, Mullins, F. H. Murray, Newman, Parsons, Pfeiffer, Pierpont, Post, Rawles, Reddick, M. S. Rees, R. G. D. Richardson, Ritt, Robertson, Seely, Serghiesco, Simons, Smail, P. A. Smith, J. M. Thomas, J. L. Walsh, Weinstein, Weisner, Whittemore, H. B. Williams, W. J. Willis, W. A. Wilson, Zippin.

There was no meeting of the Council or of the Trustees of the Society.

Associate Secretary Dresden presided at the morning session, and Professor Whittemore in the afternoon.

At the request of the Program Committee, Professor J. L. Walsh, of Harvard University, delivered, at the beginning of the afternoon session, an address entitled *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions.* This address will appear in full in an early issue of this Bulletin.

Titles and abstracts of the other papers read at this meeting follow below. The papers of Beckenbach, Calugaréano, Douglas and Emch were read by title. Mr. Beckenbach was introduced by Professor Bray, and Dr. Calugaréano by Professor Kasner.

1. Professor J. E. Donahue: *Geometric proof of Kasner's pentagon theorem.*

In a recent paper (American Mathematical Monthly, vol. 35 (1928), pp. 352 ff.), Kasner proves (using coordinates) the following theorem: *For* *any plane pentagon P the inscribed pentagon of its diagonal pentagon and the diagonal pentagon of its inscribed pentagon are identical.* In the present paper there is given a synthetic proof employing Pascal's line and Brianchon's point.

2. Miss Edna E. Kramer: *The Laguerre group and allied topics.*

It is well known that the Laguerre group can be represented analytically by the linear fractional transformations of the variable $w=u+jv$, where $j^2=0$ and u and v are functions of the Hessian coordinates of a directed line. Using this representation, the present paper deals with the Laguerre group in a manner analogous to that in which the inversion group is often treated. Some new properties of Laguerre inversion are discussed, and a large family of anallagmatic curves is found. The invariants and path-curves of the groups of integral linear transformations are discussed. For $W=w+b$ these curves are cycles (directed circles); for $W=aw$ each curve (non-oriented) is a tractrix. For the direct transformations of the Laguerre group it is found that there may be 0, 2, ∞^1, or $2\infty^1$ fixed rays in the finite plane, and for the indirect transformations, 1, 2, ∞^1 or $2\infty^1$. The transformations resulting from combinations of the integral linear transformations with reciprocation are classified kinematically. In the second part of the paper a theory of polygenic functions of w is developed, analogous to that of Kasner for polygenic functions of z, and points of similarity and divergence between the two theories are noted.

3. Professor Edward Kasner: *The biharmonic functions of Poincaré related to functions of two complex variables.*

Poincaré applies the term biharmonic to those functions of four real variables $F(x, y, x', y')$ which can be regarded as the real part of an analytic function of two complex variables $z=x+iy$, $z'=x'+iy'$. If a conformal substitution $x'=\phi(x, y)$, $y'=\psi(x, y)$ is performed, F obviously becomes a harmonic function of x, y. In the present paper two easy converses of this theorem are proved. (I) Biharmonic functions are the only functions of four real variables which are converted into harmonic functions by *every* conformal substitution. (II) The only substitutions which convert *all* biharmonic functions into harmonic functions are the conformal substitutions. Connection is made with the author's geometric representation of functions of two complex variables, first by points in four dimensions, then by bipoints or point-pairs in the plane. See abstracts in this Bulletin, vol. 15 (1908–09), pp. 67, 159.

4. Mr. Eli Gourin: *On the irreducible polynomials in several variables which become reducible when the variables are replaced by powers of themselves.*

Given an irreducible polynomial $Q(x_1, \cdots, x_p)$, consisting of more than two terms, the sets of positive integers $t_1, \cdots, t_p$ for which $Q(x_1^{t_1}, \cdots, x_p^{t_p})$ is reducible can be grouped into a finite number of classes of sets. In each class, the sets are of the form $a_1T_1, \cdots, a_pT_p$, where the T's are elements of a certain set, having the required property, and the a's are arbitrary positive integers. Those of the T's which are distinct from unity are each equal to one and the same prime number p. If M denotes the largest among the m_i's, where m_i is the degree of Q in x_i, and if P is the largest prime number which does not exceed M^2, then $p \leqq P$.

5. Professor J. F. Ritt: *Algebraic combinations of exponentials.*

This paper investigates functions defined by an equation

$$(1) \qquad \sum_{i=0}^{n} (a_{0i} + a_{1i}e^{\alpha_1 x} + \cdots + a_{mi}e^{\alpha_m x})y^i = 0,$$

where the α's and a's are any complex constants. The theory of these functions contains in itself the theory of algebraic functions. It is shown that, for large x's, every branch of a function defined by (1) can be represented by Dirichlet series with complex exponents. It is then proved that if (1) defines more than a single analytic function, the first member of (1) can be factored into expressions similar to itself but of lower degree in y. In particular, a uniform function defined by (1) must satisfy an equation of the first degree, and hence is the quotient of two exponential polynomials.

6. Professor J. F. Ritt: *Representation of analytic functions as infinite products.*

Let $f(z)$ be analytic, and equal to unity, at $z=0$. It is proved in this paper that $f(z)$ admits, for the neighborhood of the origin, a representation as an absolutely convergent product $(1+a_1z)(1+a_2z^2) \cdots (1+a_nz^n) \cdots$, with constant a's. The representation is unique.

7. Professor W. A. Wilson: *On linear upper semi-continuous collections of continua.*

Let M be a bounded upper semi-continuous collection of continua $\{X\}$ such that no two have common points and $X=f(t)$ is upper semi-continuous in the interval $a \leqq t \leqq b$; a collection of this kind may be called *linear* and M the *image* of ab. If there is no upper semi-continuous aggregate function $Y=g(t)$, where each Y is a continuum defined in ab such that $g(t) \subset f(t)$ at every point and $g(t) \neq f(t)$ at some point, $f(t)$ is called a minimal upper semi-continuous function. It was shown in a previous communication that, if $f(t)$ is a minimal upper semi-continuous function in ab, M is a bounded plane continuum, and no $f(t)$ separates $f(a)$ from $f(b)$, then M is irreducible between $f(a)$ and $f(b)$. The present paper discusses the case that the set of points $T'=\{t'\}$ for which $f(t')$ separates $f(a)$ from $f(b)$ is not void. It is shown that the conclusion of the quoted theorem remains valid if the set T' is totally disconnected, and that it is sometimes valid and sometimes not if T' contains an interval. Finally, there exist cases where $f(t)$ is a minimal upper semi-continuous function defined in an interval ab and the image of ab fills a portion of the plane.

8. Dr. O. E. Glenn: *A theory of integral invariants.*

The general question studied in this paper is as follows. A curve C: $f(x, y)=0$, is transformed by $x_1=\sigma(x, y)$, $y_1=\tau(x, y)$ into C_1: $g(x_1, y_1)$.

Suppose any function ϕ of x, y, y' to be given. It is required to find the line of integration C such that $\int_\alpha^\beta \phi(x_1, y_1, y_1')dx = M\int_\gamma^\delta \phi(x, y, y')dx$, the line of integration being C_1 in the first integral and C in the second. The limits (α, β) are to be arbitrary, and (γ, δ) are limits transformed from (α, β) by $x_1 = \sigma$, $y_1 = \tau$. We also consider the problem where ϕ is to be determined, C being assigned.

9. Dr. G. Calugaréano: *Polygenic functions considered as integrals of differential equations.*

In this paper, which will appear in the Transactions of this Society, the author applies polygenic (that is, non-analytic) functions to the solution of certain ordinary differential equations of nth order. The first derivative is defined as in Kasner's papers (Science, vol. 66 (1927), p. 981; Proceedings of the National Academy, vol. 13 (1928, p. 75). The second and higher derivatives are the special case which Kasner describes as rectilinear (Transactions of this Society, vol. 30 (1928), pp. 803–818). By means of the polygenic solutions, analytic solutions may also be obtained. The discussion is purely analytic, as in the author's Paris thesis (Nov., 1928).

10. Professor Arnold Emch: *On the mapping of the involutorial G_4 in a plane upon a Steiner surface.*

This paper appears in full in the present issue of this Bulletin.

11. Dr. Jesse Douglas (National Research Fellow): *Solution of the problem of Plateau.*

In a paper presented to the Society, Dec. 28, 1926, the author reduced the problem of Plateau to the integral equation (I): $*\int_\Gamma K(t, \tau)$ ctn $(\frac{1}{2}[\phi(\tau) - \phi(t)])d\tau = 0$ (or its equivalent by the substitution ctn $(\frac{1}{2}\phi) = \psi$: $*\int_\Gamma K(t, \tau)d\tau/[\phi(\tau) - \phi(t)] = 0$). For the notation see the abstract in this Bulletin, vol. 33, p. 143. In a subsequent paper presented April 6, 1928 (see this Bulletin, vol. 34, p. 405) the author showed that (I) expressed the vanishing of the functional derivatives of the functional

$$A(\phi) = -\int_\Gamma \int_\Gamma K(t, \tau) \log \sin \tfrac{1}{2} |(\phi(\tau) - \phi(t)| dt d\tau.$$

It thus becomes a question of proving the existence of a minimizing function ϕ for $A(\phi)$. This is accomplished in the present paper. It is shown that $A(\phi)$ is a lower semi-continuous function on the set of all functions ϕ which, by means of $\theta = \phi(t)$, establish a one-one continuous correspondence between the contour Γ and the unit circle C; only these functions are admissible. In order to have a closed set $[\phi]$ it is necessary to include functions ϕ whose cartesian t, θ graph presents (1) horizontal or (2) vertical line segments as well as (3) properly monotonically increasing continuous functions ϕ. The set $[\phi]$ is compact as well as closed, and we can therefore apply the results of Fréchet's thesis (Palermo Rendiconti, vol. 22 (1906)) to affirm the existence of a minimizing ϕ. It is shown that, for a function ϕ of type (1), $A(\phi) = +\infty$, and for one of type (2), $A'(\phi, t)$ cannot be identically zero. Hence the existing minimizing ϕ is of the proper type (3), and the problem is solved.

12. Mr. E. F. Beckenbach: *An inequality for definite Hermitian determinants.*

This paper appears in full in the present issue of this Bulletin.

13. Dr. T. H. Gronwall: *Straight line geodesics in Einstein's parallelism geometry.* Preliminary communication.

This paper gives a set of partial differential equations forming the necessary and sufficient conditions that, in Einstein's parallelism geometry, all geodesics be straight lines and vice versa.

14. Dr. T. H. Rawles: *On the inverse problem in the calculus of variations.*

The problem considered in this paper is that of finding the integrand function which is minimized by a given two-parameter family of extremals. By approaching the problem from the standpoint of the invariant integral a method is obtained which is simpler than that of Darboux in that it is not necessary to solve a partial differential equation.

15. Professor James Pierpont: *Foucault's pendulum in elliptic space.*

The equations of motion are deduced from Hamilton's principle. We find $2\dot{\psi}(\dot{\omega}+kp)+\psi\ddot{\omega}=k^2p'\psi \sin\omega\cos\omega-k^2pp'(r/l)\sin\omega$, where ω is the azimuth of the plane of vibration, ψ the small angle that the pendulum makes with the vertical, k the angular velocity of the earth, ϕ the latitude of the station, $p=\sin\phi$, $p'=\cos\phi$, $l=\sin L$, $r=\sin\rho$, L being the length of the pendulum, ρ being the distance of the point of suspension from the center of the earth, both expressed in elliptic measure. The space constant is taken as unity. The above equation is entirely analogous to the corresponding equation in classical mechanics. Under similar conditions we may say, therefore, that in first approximation the angular velocity of the plane of vibration is $\dot{\omega}=-k\sin\phi$. The author wishes to note that the analysis employed may be readily extended to hyperbolic space.

16. Dr. T. H. Gronwall: *On the differential equation of the vibrating membrane.*

This paper considers the question of the analytic continuation of a solution of the partial differential equation $\nabla^2u+k^2u=0$ (with boundary condition $u=0$ or $\partial u/\partial n=0$) across a straight line part of the boundary. The result is applied to the determination of the frequencies and characteristic functions for a rectangular membrane.

17. Mr. Leo Zippin: *A characterization of the simple closed surface and of the plane.*

It is proved that a continuous curve C, such that (i) any (1, 1) bicontinuous transformation of a simple closed curve of C into itself or another simple closed curve of C can be extended to C, and (ii) there is at least one simple closed curve of C which disconnects it, is a simple closed surface if C is bounded and homeomorphic with the plane if C is unbounded.

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Secretary.