ON THE MAPPING OF THE QUADRUPLES OF THE INVOLUTORIAL G_4 IN A PLANE UPON A STEINER SURFACE*

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1. *Introduction.* Castelnuovo has definitely shown† that every plane involution may be mapped uniformly upon a rational surface. As may be expected, and as the author has shown in case of the involution of sextuples,‡ from such a mapping process arise interesting properties of certain configurations and curves which reflect geometric properties of the corresponding surface, and conversely. In this paper the involution induced by the group

$$G_4 \equiv \begin{pmatrix} \pm x_1, & \pm x_2, & \pm x_3 \\ x_1, & x_2, & x_3 \end{pmatrix}$$

is investigated from this standpoint.

In what follows I shall denote the involutorial quadruple in the plane (x) merely by G_4. To construct this, let $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$ be the coordinate triangle, and $B(x_1, x_2, x_3)$ a generic point. Join B to A_1, A_2, A_3 and construct the fourth harmonic lines to BA_1, BA_2, BA_3 with respect to the pairs of sides A_1A_2, A_1A_3; A_2A_3, A_2A_1; A_3A_1, A_3A_2. These three lines intersect in the triangle $B_1(-x_1, x_2, x_3)$, $B_2(x_1, -x_2, x_3)$, $B_3(x_1, x_2, -x_3)$, which together with $B(x_1, x_2, x_3)$ form the G_4. If we construct for each B_i the symmetric G_6 and denote it by (B_i), we obtain the configuration of the octahedral group

$$G_{24} \equiv \begin{pmatrix} \pm x_i, & \pm x_k, & \pm x_l \\ x_1, & x_2, & x_3 \end{pmatrix},$$

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† *Sulla razionalità delle involuzioni piane,* Mathematische Annalen, vol. 44 (1894), pp. 125–155.

‡ *On the mapping of the sextuples of the symmetric substitution group G_6 in a plane upon a quadric,* this Bulletin, vol. 33 (1927), pp. 745–750.

consisting of the four sextuples (B), (B_1), (B_2), (B_3), which lie each on a conic as is well known.*

2. *Mapping on the Steiner Surface.* The substitutions of the G_4 outside of identity, expressed as collineations, are the three involutorial perspectives

$$I_1 \equiv (\rho x_1' = -x_1, \quad \rho x_2' = x_2, \quad \rho x_3' = x_3),$$
$$I_2 \equiv (\rho x_1' = x_1, \quad \rho x_2' = -x_2, \quad \rho x_3' = x_3),$$
$$I_3 \equiv (\rho x_1' = x_1, \quad \rho x_2' = x_2, \quad \rho x_3' = -x_3),$$

which leave the quadruple G_4 invariant. If we propose to construct those integral rational homogeneous functions in (x) which are invariant under these perspectives, we must form homogeneous functions of the squares of the variables: $F(x_1^2, x_2^2, x_3^2)$. The simplest set of these is evidently $a_1x_1^2+a_2x_2^2+a_3x_3^2=0$, of which type there are only three linearly independent. These conics form a net with $A_1A_2A_3$ as a common self-polar triangle. Next in order is the quartic

$$(1) \quad b_1x_1^4 + b_2x_2^4 + b_3x_3^4 + c_1x_2^2x_3^2 + c_2x_3^2x_1^2 + c_3x_1^2x_2^2 = 0,$$

which may be reduced by a collineation to the form

$$(2) \quad a_1x_2^2x_3^2 + a_2x_3^2x_1^2 + a_3x_1^2x_2^2 + a_4(x_1^4 + x_2^4 + x_3^4) = 0,$$

so that all quartics of the webb (2) may be expressed linearly by the relations

$$(3) \quad \begin{aligned} \rho y_1 = x_2^2x_3^2, \quad \rho y_2 = x_3^2x_1^2, \quad \rho y_3 = x_1^2x_2^2, \\ \rho y_4 = x_1^4 + x_2^4 + x_3^4. \end{aligned}$$

From (3) it is evident that to the quadruple G_4 in (x) corresponds in the space (y) uniquely a point. The locus of these points is easily obtained as the Steiner surface Γ_4

$$(4) \quad y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2 - y_1y_2y_3y_4 = 0.$$

Hence we may state the following theorem.

* Emch, A., *Some geometric applications of symmetric substitution groups*, American Journal of Mathematics, vol. 45 (1923), pp. 192–207.

THEOREM 1. *By the transformation* (3) *the Steiner surface* (4) *is mapped upon a plane, so that to a generic point on the surface correspond four points B, B_1, B_2, B_3 of a self-projective quadruple in the group of involutions I, I_1, I_2, I_3.*

3. *Properties of Quadruples.* To a plane section of Γ_4 by $a_1y_1+a_2y_2+a_3y_3+a_4y_4=0$ corresponds in (x) the quartic (2). In particular to a section of Γ_4 by a plane through the triple-point A_4 of Γ_4 corresponds the quartic

$$a_1x_2^2x_3^2 + a_2x_3^2x_1^2 + a_3x_1^2x_2^2 = 0, \tag{5}$$

and to the section by $y_4=0$ the quartic

$$x_1^4 + x_2^4 + x_3^4 = 0. \tag{6}$$

The quartic (5) is the transform of the conic

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \tag{7}$$

by the quadratic transformation $\rho x_1' = x_2x_3$, $\rho x_2' = x_3x_1$, $\rho x_3' = x_1x_2$. To the conic (7) corresponds on Γ_4 a conic which is the residual intersection by the quadric $a_1y_2y_3+a_2y_3y_1+a_3y_1y_2=0$. When a point B lies on (7), then all points of the quadruple G_4 determined by it lie on it. Now two generic points determine the conic (7), so that we have the following obvious theorem.

THEOREM 2. *Two arbitrary distinct quadruples G_4 lie on a definite conic of the net* (7).

A tangent plane at a generic point P of Γ_4 cuts the surface in two conics K_1 and K_2, whose four intersections lie at P and on A_4A_1, A_4A_2, A_4A_3. To these conics correspond in (x) two conics on the quadrangle corresponding to P. Obviously, any conic in (x) may be one of these conics. The product of the two conics in (x), corresponding to K_1 and K_2, must have the form (2). Hence when $a_1x_1^2+a_2x_2^2+a_3x_3^2=0$ is one of the conics, the other $b_1x_1^2+b_2x_2^2+b_3x_3^2=0$ must be such that their product will have the form (2). This is the case when $b_1=1/a_1$, $b_2=1/a_2$, $b_3=1/a_3$, so that their product,

$(a_1x_1^2+a_2x_2^2+a_3x_3^2)(a_2a_3x_1^2+a_3a_1x_2^2+a_1a_2x_3^2)=0$, when expanded in the form (2), becomes

$$(8)\quad a_1(a_2^2+a_3^2)x_2^2x_3^2+a_2(a_3^2+a_1^2)x_3^2x_1^2 + a_3(a_1^2+a_2^2)x_1^2x_2^2+a_1a_2a_3(x_1^4+x_2^4+x_3^4)=0.$$

The plane to whose section with Γ_4 corresponds in (x) the degenerate quartic (8) is therefore

$$(9)\quad a_1(a_2^2+a_3^2)y_1+a_2(a_3^2+a_1^2)y_2 + a_3(a_1^2+a_2^2)y_3+a_1a_2a_3y_4=0.$$

As the coordinates of this plane are $\rho u_1=a_1(a_2^2+a_3^2)$, $\rho u_2=a_2(a_3^2+a_1^2)$, $\rho u_3=a_3(a_1^2+a_2^2)$, $\rho u_4=a_1a_2a_3$, the envelope of the tangent-planes of the Steiner surface, by elimination of the a's, is easily obtained as

$$(10)\qquad 4u_4^3-(u_1^2+u_2^2+u_3^2)u_4+u_1u_2u_3=0,$$

a cubic of class 3, as is well known.

Consider now a generic line g in (y) cutting Γ in four points P_1, P_2, P_3, P_4. To these correspond in (x) four quadruples G_4^1, G_4^2, G_4^3, G_4^4, which together form the base points of the pencil of quartics corresponding to the plane sections through g. Through g we can draw three tangent planes to Γ_4 according to (10), which touch Γ_4 in T_1, T_2, T_3. To these T's correspond in (x) three quadruples D_1, D_2, D_3. The tangent planes t_1, t_2, t_3 cut Γ_4 in three pairs of conics, of which each passes through the four P's and one of the points T, and to which correspond in (x) also three pairs of conics. Each of the latter contains all four quadruples G_4^i, $i=1, 2, 3, 4$, and one of the quadruples D. Hence the quadruples G_4 and D are as entities precisely in the relation of a complete quadrangle with its three diagonal points.

The analogy of the quadruples G and D with a complete quadrangle leads to a very simple geometric proof that the Steiner surface is of class 3. Through the quadruples G we can pass three pairs of conics $K(G_4^1, G_4^2)$, $K(G_4^3, G_4^4)$; $K(G_4^1, G_4^3)$, $K(G_4^2, G_4^4)$; $K(G_4^1, G_4^4)$, $K(G_4^2, G_4^3)$, such that each pair contains all four quadruples. Each of these pairs has a quadruple D as base points, and to each corresponds on Γ_4 two conics through P_1, P_2, P_3, P_4 and one of the points T. The plane of these conics touches Γ_4 at T. Hence there are three such tangent-planes through g to the Steiner surface. These results may be stated in the following form.

THEOREM 3. *The quadruples of the involutorial G_4 in the plane (x) may be arranged in groups of seven according to a complete quadrangle array. Any two quadruples in (x) determine such a configuration. Thus there are ∞^4 such configurations. To each such configuration correspond on the Steiner surface the four intersections of a line g with Γ_4 and the three points of tangency of the tangent-planes from g to Γ_4.*

4. *Quadruples on Quartics.* To a point on Γ_4:

$$y_1,\ y_2,\ y_3,\ (y_2^2y_3^2 + y_3^2y_1^2 + y_1^2y_2^2)/(y_1y_2y_3),$$

corresponds in (x) the quadruple

$$[(y_2y_3)^{1/2},\ (y_3y_1)^{1/2}, (y_1y_2)^{1/2}]\ ;\ [-(y_2y_3)^{1/2}, (y_3y_1)^{1/2},\ (y_1y_2)^{1/2}]\ ;$$
$$[(y_2y_3)^{1/2}, -(y_3y_1)^{1/2}, (y_1y_2)^{1/2}]\ ;\ [\ (y_2y_3)^{1/2}, (y_3y_1)^{1/2}, -(y_1y_2)^{1/2}].$$

To a point on $x_1=0$ corresponds on Γ_4, $\rho y_1=x_2^2x_3^2$, 0, 0, $\rho y_4=x_2^4+x_3^4$, that is, a definite point on A_4A_1 in (y). To this same point, corresponds in (x) a quadruple which lies on $x_1=0$, and which, from the form of the equation $x_2^4-(y_1/y_4)x_2^2x_3^2+x_3^4=0$, is easily seen to have the form $(0, x_2, x_3)$; $(0, x_2, -x_3)$; $(0, x_3, x_2)$; $(0, x_3, -x_2)$. The third and fourth of these points are obtained by projecting the inverse of (x_1, x_2, x_3) and $(x_1, x_2, -x_3)$ in the quadratic involution $x_i'=1/x_i$ from A_1 in (x) upon $x_1=0$. The first two points, each counted twice, must be considered as a degenerate involutorial quadrangle. The same is true of the other two.

Now a generic line l in (x) cuts a quartic C_4 in four points; consequently to l corresponds on Γ_4 a space quartic L_4, since it cuts the curve D_4, corresponding to C_4, and its plane in four points. As l cuts each of the sides of the coordinate triangle in one point, L_4 cuts each A_4A_1, A_4A_2, A_4A_3 in (y), say in B_1, B_2, B_3. To L_4 correspond in (x) besides l the three associated lines l_1, l_2, l_3 in the involution, which together form an involutorial quadrilateral L whose vertices lie two by two on the sides of the coordinate triangle. If $a_1x_1+a_2x_2+a_3x_3=0$ is the line l, then the pole of l with respect to the coordinate triangle $A_1A_2A_3$ is $E(a_1a_3,\ a_3a_1,\ a_1a_2)$ and its quadratic transform $F(a_1,\ a_2,\ a_3)$. The triangular polar of F is $m=a_2a_3x_1+a_3a_1x_2+a_1a_2x_3=0$. This determines another involutorial quadrilateral $M=mm_1m_2m_3$ in (x) whose vertices lie again by pairs on the sides of $A_1A_2A_3$. To these vertices correspond on Γ_4 the same points B_1, B_2, B_3 as in the case of L. To the quadrilateral M corresponds on Γ_4 a space quartic M_4 through $B_1B_2B_3$. The plane β of these points cuts Γ_4 in a quartic B_4, and L_4 and M_4 each cut B_4 in a fourth point outside of B_1, B_2, B_3. The plane β has the form

$$\begin{aligned}\beta = a_1^2(a_2^4 + a_3^4)y_1 + a_2^2(a_3^4 + a_1^4)y_2 \ .\\ + a_3^2(a_1^4 + a_2^4)y_3 - a_1^2a_2^2a_3^2y_4 = 0.\end{aligned} \tag{11}$$

Denoting the coordinates of this plane by u_1, u_2, u_3, u_4, its envelope is easily found as the class-cubic

$$4u_4^3 - (u_1^2 + u_2^2 + u_3^2)u_4 - u_1u_2u_3 = 0. \tag{12}$$

This is, however, different from the Steiner class-cubic. The plane β cuts Γ_4 in a quartic B_4 to which corresponds in (x) the quartic N:

$$\begin{aligned}a_1^2(a_2^4 + a_3^4)x_2^2x_3^2 + a_2^2(a_3^4 + a_1^4)x_3^2x_1^2\\ + a_3^2(a_1^4 + a_2^4)x_1^2x_2^2 - a_1^2a_2^2a_3^2(x_1^4 + x_2^4 + x_3^4) = 0.\end{aligned} \tag{13}$$

N passes through the 12 vertices of the quadrilaterals L and M in (x), and moreover through the involutorial quadrangles corresponding to the fourth intersections of L_4 and M_4 with B_4. These results may be stated as follows.

THEOREM 4. *There exists in (x) a system of quartics which cut the sides of the fundamental triangle of the involution in 12 points of two complete quadrilaterals. To these correspond in (y) two space quartics through the same three points $B_1B_2B_3$*

on the double lines of Γ_4. *The locus of the planes* β *is a certain class-cubic.*

To the general invariant quartic of the involution

$$(14)\quad b_1x_1^4 + b_2x_2^4 + b_3x_3^4 + c_1x_2^2x_3^2 + c_2x_3^2x_1^2 + c_3x_1^2x_2^2 = 0$$

corresponds on Γ_4 a space quartic cut out by the cone

$$(15)\quad b_1y_2^2y_3^2 + b_2y_3^2y_1^2 + b_3y_1^2y_2^2 + y_1y_2y_3(c_1y_1 + c_2y_2 + c_3y_3) = 0,$$

which has A_4A_1, A_4A_2, A_4A_3 in (y) as nodal lines. When $b_1 = b_2 = b_3 = c_4$, then by means of Γ_4, (15) reduces to $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$, which verifies the results in case of a plane quartic on Γ_4.

Concerning the base points of a pencil of quartics of the general involutorial type we may state the following fact.

THEOREM 5. *The* 16 *base points of a pencil of involutorial quartics in* (x) *form* 4 *involutorial quadruples which lie two by two on* 6 *involutorial conics.*

5. *Flexes and Double Tangents.* If we draw the three tangent-planes from a line g to Γ_4, they cut Γ_4 in three pairs of conics through $(P_1P_2,\ P_3P_4)$; $(P_1P_3,\ P_2P_4)$; $(P_1P_4,\ P_2P_3)$. With every couple P_iP_k is associated just one such conic, just as in (x) there is just one conic through the quadruples G_4^i and G_4^k. As two of the points P determine the remaining two on the line joining them, it is evident that *there is just one such conic through two generic points* P_i *and* P_k *on* Γ_4.

A generic plane cuts Γ_4 in a rational quartic to which corresponds in (x) a quartic C_4 without double points. This is clear, since to a double point of L on A_4A_1 in (y) correspond in (x) four distinct points on A_2A_3, and to every other point of L corresponds a proper involutorial quadruple on C_4. The Hessian of C_4 is a sextic H_6 which as a covariant of C_4 cuts C_4 in 6 involutorial flex-quadruples. Hence to H_6 corresponds on Γ_4 a sextic which cuts the quartic L, corresponding to C_4, in six points. These form 15 couples P_iP_k and through each of these couples passes a conic on Γ_4. Hence we may state the following theorem.

THEOREM 6. *The 24 flexes of a quartic C_4 in (x) lie on 15 conics, of which each contains 8 of the 24 flexes.*

This follows, of course, also from the fact that any two generic involutorial quadruples lie on a conic.

For a plane of the pencil $y_1+y_2+y_3+\lambda y_4=0$, the corresponding quartic C_4 in (x) becomes symmetric, so that with every point on C_4 are associated 5 other points which together lie on a conic. *Hence in this case, in addition to the 15 conics of the flexes there are 4 other conics of which each contains 6 of the flexes.*

The 28 double tangents of C_4 may be grouped in 7 involutorial quadrilaterals, each with two involutorial quadrangles of points of tangency, so that the 56 points of tangency form 14 involutorial quadrangles.

THEOREM 7. *The 28 double tangents of the C_4 form 7 involutorial quadrilaterals whose 56 points of tangency lie 8 by 8 on $\binom{14}{2}=91$ conics.*

6. *Tropes.* There are four tangent planes touching Γ_4 along conics. To these correspond in (x) four couples of coincident conics, which must have the form $(a_1x_1^2+a_2x_2^2+a_3x_3^2)^2=a_1^2x_1^4+a_2^2x_2^4+a_3^2x_3^4+2\,(a_1a_2x_1^2x_2^2+a_2a_3x_2^2x_3^2+a_3a_1x_3^2x_1^2)=0$. But for a quartic corresponding to a plane section there must be $a_1^2=a_2^2=a_3^2$, which for (a_1, a_2, a_3) gives the possible solutions (a, a, a); $(-a, a, a)$; $(a, -a, a)$; $(a, a, -a)$. Consequently the four tropes of the Steiner surface are

$$(16)\qquad \begin{aligned} 2(y_1+y_2+y_3)+y_4&=0,\\ 2(y_1-y_2-y_3)+y_4&=0,\\ 2(-y_1+y_2-y_3)+y_4&=0,\\ 2(-y_1-y_2+y_3)+y_4&=0. \end{aligned}$$

7. *Quartics on G_{48}.* When (x) and (x') are connected by the quadratic involution $x_i'=1/x_i$, then the corresponding points $P(y)$ and $P'(y')$ on the map Γ_4 are also birationally connected. The corresponding involution on Γ_4 has the form

$$\delta y_1' = y_2 y_3, \quad \delta y_2' = y_3 y_1, \quad \delta y_3' = y_1 y_2, \quad \delta y_4' = y_1'^2 + y_2'^2 + y_3'^2, \tag{17}$$

considering that $y_4 = (y_2^2 y_3^2 + y_3^2 y_1^2 + y_1^2 y_2^2)/(y_1 y_2 y_3)$. If we form the line coordinates $p_{ik} = y_i y_k' - y_k y_i'$ of PP', we obtain a rational congruence, whose intersection with a complex is a ruled surface which cuts Γ_4 in an invariant curve of the involution $(y) \rightleftarrows (y')$. To these correspond in (x) invariant curves of the involution $(x) \rightleftarrows (x')$. Such curves are for example the sextics

$$x_1^2(x_2^2 - x_3^2)^2 + x_2^2(x_3^2 - x_1^2)^2 + x_3^2(x_1^2 - x_2^2)^2 = 0, \tag{18}$$

which is the locus of couples of corresponding points (x), (x') on the tangents of the class-conic $u_1^2 + u_2^2 + u_3^2 = 0$, and

$$x_1^2(x_2^2 + x_3^2)^2 + x_2^2(x_3^2 + x_1^2)^2 + x_3^2(x_1^2 + x_2^2)^2 + \lambda x_1^2 x_2^2 x_3^2 = 0. \tag{19}$$

Both are invariant in the Cremona group G_{48}, which is the product of the quadratic involution and the octahedral group G_{24}.

The curve (18) has the vertices of the fundamental triangle and the four invariant points of the quadratic involution as double points and is of the genus three type.

8. *Clebsch and Lüroth Curves.* The quartics which are invariant under the octahedral group G_{24} have the form

$$\Sigma x_i^4 + 6\lambda \Sigma x_i^2 x_k^2 = 0. \tag{20}$$

Some of these curves are rather well known.* The covariant S is in the pencil, and the parameter λ for S has the value $\lambda' = (\lambda^2 - 2\lambda^3 - \lambda^4)/(6\lambda^4)$. The question whether there are any Clebsch and Lüroth curves in the pencil can easily be answered by determining those values of λ for which the so-called Clebsch determinant of order 6 vanishes.

In case of the quartic (20), this determinant reduces to the form

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* Ciani, E., *I varî tipi possibili di quartiche piane piu volte omologico-armoniche*, Rendiconti di Palermo, vol. 13 (1899), pp. 347–373.

$$(21) \qquad \lambda^3 \begin{vmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{vmatrix} = 0.$$

The roots are $\lambda=0$, $\lambda=1$, $\lambda=-1/2$. The first two roots are three- and two-fold, respectively. For $\lambda=0$ we obtain the Dyck curve for which the covariant S becomes indeterminate. For $\lambda=1$ the apolar conic has the form $a(u_1^2-u_2^2)-b(u_2^2-u_3^2)=0$, and (20) becomes

$$C_4 = (x_1+x_2+x_3)^4+(-x_1+x_2+x_3)^4 + (x_1-x_2+x_3)^4+(x_1+x_2-x_3)^4 = 0,$$

$$S = (x_1+x_2+x_3)(-x_1+x_2+x_3) \cdot (x_1-x_2+x_3)(x_1+x_2-x_3) = 0.$$

For $\lambda=-1/2$, the apolar conic simply assumes the form $u_1^2+u_2^2+u_3^2=0$, and

$$C_4 = x_1^4+x_2^4+x_3^4-3(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)=0,$$

$$S = x_1^4+x_2^4+x_3^4+7(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)=0.$$

These Clebsch and Lüroth curves invariant under the octahedral collineation group seem to be new.

The construction of these curves, outside its tediousness, does not present particular difficulties. Any tangent t_1 of the apolar conic cuts S in four points from which four other tangents t_2, t_3, t_4, t_5 may be drawn. These together with the first form one of the ∞^1 pentagons inscribed to the Lüroth curve. By means of such a pentagon both curves may be represented in the well known manner:

$$C_4 = \Sigma a_i t_i^4 = 0, \qquad S = \Sigma b_{ijkl} t_i t_j t_k t_l = 0.$$

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