either explicitly or tacitly, that demand is a function of price, adequately represented by the notation $D=f(p)$. Cournot is also not alone in considering profit as the central quantity in economics, whose maximum determines the other quantities; and given the industrial habits of his time, he is hardly to be blamed.

The translation seems to be carefully done, and the editor has added numerous notes designed to assist the less mathematical reader. No attempt has been made to bring the bibliography down to the date of this edition; in the thirty years since the treatise first appeared in English the additions to the bibliography included in that edition would presumably have surpassed the desirable limits for the book. There would have been service rendered, however, if the editor had used the resources at his disposal to clear up the ambiguous character of some of the references already given. This is not said to diminish the gratitude which the public owes to both editor and translator in making again accessible, after ninety years, the eternal freshness of this brief classic.

G. C. EVANS

THREE BOOKS ON NON-EUCLIDEAN GEOMETRY

Vorlesungen über Nicht-Euklidische Geometrie. By Felix Klein. Prepared anew for publication by W. Rosemann. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 26. Berlin, Julius Springer, 1928. xii+326 pp. Price 18 marks unbound, 19.50 marks in linen.

Nicht-Euklidische Geometrie. Hyperbolische Geometrie der Ebene. By Richard Baldus. Sammlung Göschen, No. 970. Berlin and Leipzig, Walter de Gruyter, 1927. 152 pp. Price 1.50 marks.

La Géométrie Non-Euclidienne. By P. Barbarin. Third edition, followed by notes by A. Buhl. Collection "Scientia," No. 15. Paris, Gauthier-Villars, 1928. 176 pp.

In a note to his Erlanger Programm, Klein justified his avoidance of the word "non-euclidean" in that address, saying: "There are connected with the designation 'non-euclidean geometry' a number of unmathematical conceptions which occasion as much enthusiasm on the one side as horror on the other, but which have nothing whatever to do with our purely mathematical considerations." This may be a reason why, aside from a few glances forward, that same word is absent from over one half of his admirable treatise, *Vorlesungen über Nicht-Euklidische Geometrie.* In fact, the reader with a purely euclidean past will be led so tactfully and persuasively from his usual geometry to the more general one that he may long wonder wherein lies the book's right to its title. He will be an admirer of the greater beauty and symmetry of non-euclidean space without realizing how far he has left euclidean space behind him.

Part one of the book is an introduction to projective geometry. We pass rapidly from elementary affine coordinates in euclidean space to homogeneous and general projective coordinates. Thus, the arithmetical basis of cartesian geometry being assumed, no time need be given to axioms. Analysis situs enters early, when the projective plane is shown to be one-sided. The chapter on fundamental concepts treats, further, of homogeneous linear substitutions and projective transformations (the Erlanger Programm is to be our guide), n-dimensional manifolds, duality, cross ratios, imaginary elements. There follows a careful treatment of configurations of second degree (both class and order), a complete classification from the point of view of reality being given. A number of transitions from one non-degenerate form to another through a degenerate one are traced—such, for instance, as will later clarify the position of euclidean geometry between its two rivals. The last chapter on projective geometry deals with those collineations which leave invariant a configuration (degenerate or otherwise) of second degree. The fate of points, the families of invariant surfaces, the interpretation of real collineations as motions,—rotations, translations, screw-motions—are studied clearly and fully, for various types of fundamental configurations.

The second part deals with measurement, on a projective basis. Again taking thought that the reader begin with the familiar, Klein discusses the euclidean formulas for lengths and angles, and shows their projective relation to the circular points (or fundamental circle, if there are three dimensions). Then the metric of the bundle of lines or planes (e.g., in euclidean space) gives the first glimpse of measurement in elliptic and spherical geometries. It is noted, as a matter of interest, that elliptic geometry prevails on the euclidean plane at infinity. Only in the fifth chapter is a projective coordinate system set up in its own right, by the familiar repetition of the harmonic construction. A notable point in Klein's treatment is his insistence that coordinates can be set up by constructions entirely within a finite region. The way is then clear for metrical definitions on the basis of the absolute (first non-degenerate, then degenerate) by means of the logarithm of a cross ratio. Now a selection has to be made, among all the possible fundamental configurations, in order that angular measurement be elliptic; and there result, of course, the three ordinary types. The development of their distinctive properties is full and clear. Among the sections of unusual interest may be mentioned the equation, valid in all three geometries, between the dihedral and the trihedral angles of a tetrahedron and the analogous equation in n dimensions; the use of quaternions to represent motions in elliptic space; the excellent study of the Clifford surface and its euclidean geometry. Chapter IX, the last one of this part, discusses those forms of space which admit of elliptic, hyperbolic, euclidean metric. This discussion of analysis situs will probably be among the most influential sections in clarifying geometrical ideas. In the last paragraph we read: "From the assumption that the space about us displays a euclidean or a hyperbolic structure, it by no means follows that this space has infinite extent."

Chapter X is concerned with history, with differential geometry, and with a variety of conformal representations of non-euclidean planes on the euclidean. Finally, a few pages are given to applications of non-euclidean geometry—the use of hyperbolic motions in the study of linear substitutions of a complex variable and in that of automorphic functions, the use of non-euclidean geometry in topological study, and of projective metric in special relativity. These chapters are brief—probably no fault, for the book has pictured the subject so alluringly that the references here given should lead many students on to wider reading than one treatise could furnish.

There is scarcely an improvement which the reviewer could suggest, even in such a small matter as proof-reading. One sentence, on page 26, with regard to those collineations of a line which result from two real projections, seemed at first reading self-contradictory: "Die so erhaltenen Kollineationen besitzen stets einen reellen Fixpunkt, der aber · · · im allgemeinen nicht vorhanden zu sein braucht." Clarity results if we change to "in den allgemeinen Kollineationen." On page 116, line 9, we should read "nullteiligen und ringartigen Flächen," not "nullteiligen und ovalen Flachen"—the only error of statement noted.

It is surprising that the two words "complex" and "imaginary" are used indiscriminately, whether real numbers are included in the domain intended or excluded from it. A few random cases are the following:

Page 73, line 4, "imaginary" collineations may be real.
Page 93, line 21, "complex" collineations may be real.
Page 93, line 24, "imaginary" point-pairs may not be real.
Page 100, footnote, "complex" values may not be real.
Page 185, line 15, "imaginary" region includes real elements.
Page 236, line 13, "imaginary" region includes real elements.
Page 236, line 18, "imaginary" lines may not be real.

Are not imaginary numbers those complex numbers which are not real?

A similar oversight occurs when Clifford parallels are discussed. On page 234 we read "A Clifford parallel to a line a is never in a plane with a, but always skew to it." In order that this be true, it is necessary that parallels be real. Yet, on page 237, imaginary lines, generators of the fundamental null surface, are made members of a family of parallels, and we have therefore intersecting Clifford parallels. Perhaps it would be best not to name each of the two displacements whose product is a general motion a "Schiebung langs einer *Parallelenschar.*"

It may be due to a desire for brevity that, although curvature is carefully explained (p. 280), torsion is soon afterwards mentioned without definition (p. 305), and that a number of important theorems are not proved —for instance, that of the uniqueness of the common self-conjugate triangle of two conics. These are very slight departures from a fulfillment of the evident desire to write a book comprehensible immediately after a first course in calculus.

The value of the treatise, great in any case because of its elegance and comprehensiveness, is enhanced by a profusion of historical notes, by a generous use of cross references, looking both forward and backward, and by such a large collection of clear illustrations as no other work on non-euclidean geometry approaches. A few suggestions for future research are given. Thus: "a closed three-dimensional hyperbolic space-form of finite volume does not seem to have been found as yet"; "surfaces [of constant curvature] are fixed by difficult differential equations, which have as yet been solved only in particularly simple special cases."

The work has been prepared for the press by Rosemann. As the editor states in the preface, he gave it years of unselfish labor, first in consultation with Klein, then alone. That there resulted such a valuable contribution to mathematical literature is cause for deep gratitude to him.

Although the book of Baldus on non-euclidean geometry, in the Sammlung Göschen, is much smaller, he has, by restricting himself to hyperbolic plane geometry, been able to give an excellent presentation.

The theory common to euclidean and hyperbolic geometries is developed as far as possible, largely on the same basis as in Hilbert's *Grundlagen der Geometrie*. Often, indeed, the reader is referred to that book for proofs of familiar theorems. In this development, elliptic geometry is excluded by the axiom: "Among three points of a line one and only one lies between the other two." Next it is shown that the axioms preceding that of parallelism are valid if we use the unit circle as the fundamental conic, and interpret congruence by means of automorphic collineations of that circle and the preservation of the usual cross ratio. There follows the hyperbolic axiom, in the following frugal form: "There exist one line g and one point P outside of g, such that (at least) two lines through P fail to meet the line g." Angles obtain measure, in this interpretation, without appeal to the cross ratio made with imaginary lines. For this purpose, an angle whose vertex is at the center is given its euclidean measure, while the hyperbolic magnitude of another is defined as that resulting after a "motion" has brought the vertex to the center. Indeed, the device of placing an important point of a configuration at this center, and then applying euclidean theorems in order to obtain hyperbolic ones, is used ingeniously many times. When length has been defined as half the logarithm of the cross ratio, the material is on hand for a good treatment of trigonometry and elementary analytic geometry, of properties of lines and the various conics of area. It is interesting to note that, although the concurrence of the medians of a triangle holds in hyperbolic geometry, the proof is via euclidean geometry and that (according to Baldus) no proof independent of the parallel axiom has been discovered.

Baldus, like Klein, has a satisfactory historical account. (In fact, the reviewer recalls no other branch of mathematics in which the reader is so sure to be entertained by the wisdom and errors of the past.) Much of the booklet's value lies in the philosophic quality of the author's thought. Particularly good are the sections on the significance of "absolute" geometry and on the formalization of geometry, and his reference to the doubts which might arise from the use of a euclidean circle as the basis for hyperbolic geometry, as well as the answer to those doubts.

Figures are abundant, very clear and carefully drawn. There is a short reference list—eleven books—at the beginning. All the works named are available in German. Perhaps, if the author had realized how welcome are the worthy, inexpensive members of the Sammlung Göschen in non-German parts of the world, he would have included such important treatises as those of Coolidge and MacLeod.

The third work on non-euclidean geometry which has come to hand in these days is the third edition of Barbarin's *La Géométrie Non-Euclidienne*. The first edition, in 1902, was a slight book, following the historical line of development. Later editions, in 1907 and 1928, have been rewritten and expanded as other related disciplines developed and as Barbarin's own researches were made and published in scattered journals. Now the volume, more than double its original size, contains, also, important supplementary notes by Buhl. There results a book written, in turn, for three publics: for those who, with scarcely higher mathematics than trigonometry, would learn the essential elementary facts of these geometries; for those who are interested in the more difficult study, though with simple tools, of questions of construction and the like; and for those who are anxious to learn, from a very competent writer, of the relations between non-euclidean geometry and some recent work in physics, in analysis, and in differential geometry.

The historical pages of this book are made more interesting by a number of portraits, more valuable by the inclusion of several theorems and proofs. Since the elementary point of view, that of the searchers after a proof of the parallel postulate, is maintained, there cannot be the same elegant symmetry as projective geometry grants. On the other hand, spherical geometry (which Barbarin prefers to elliptic) and hyperbolic geometry are presented to us somewhat symmetrically. Of two postulates—namely, the sixth (that two lines cannot enclose a space) and the fifth (in Euclid's form)—each is fulfilled by one and only one of the two theories. It is worth while to emphasize that Euclid's parallel postulate is true in Riemann's geometry. Unusual emphasis is placed on de Tilly, particularly on his proof that we must have one of three identities between the coordinates of five points in space,—identities which involve the distances of all pairs of points, and lead respectively to the three geometries. Those of us who prefer the projective approach to the subject will perhaps not rank de Tilly's work as highly as Barbarin does; nevertheless it is well that stress is laid on his researches. Another seemingly excessive enthusiasm is that for Euclid's *Elements*: "Leur emploi universel des arguments les plus complets et les plus rigoureux;" "les postulats d'Euclide sont absolument rigoureux"—it is hard to say what is meant by rigor of axioms.

The other chapters which remain from the first edition summarize the main facts on lines and the various types of circles, develop trigonometric formulas of triangles and trirectangular quadrilaterals, discuss areas and volumes. It is curious that the author does not notice that, in Riemannian geometry, equidistant curves (his "hypercycles") are circles. Unfortunately, proofs are so often absent that the reader may not be convinced.

Barbarin points out that Euclid's reasoning is always by means of figures of which the construction is known. He has followed this clew, and has interested himself in the discovery of methods of construction in non-euclidean geometry. Thus, the new chapters describe a mechanism for tracing horocycles; the construction of angles of rational trigonometric functions, of line segments whose trigonometric (or hyperbolic) functions are rational, or roots of index 2^n, of rational numbers; inscription of regular polygons; the quadrature of the circle.

A chapter on the impossibility of proving the fifth postulate takes time to refute some very crude objections to the geometries of Lobachevsky and Riemann, discusses the surfaces on which their plane geometries are valid,—and, incidentally, contains a welcome photograph of Beltrami's paper pseudosphere.

With the supplementary notes of Buhl, who wrote the pamphlet on "Formules Stokiennes" of the Mémorial, we seem to begin a new book, one on those formulas. (Neither here nor in the Mémorial does he trouble to define the adjective.) The second note, starting with the derivation of Maxwell's equations from a Stokesian one, passes soon to the problem of the linear transformations leaving a homogeneous quadratic function invariant—that is, to that Cayleyan problem which was at the basis of Klein's treatise. On this basis, then, Buhl develops non-euclidean geometry anew. Twenty pages are next devoted to the much more general differential geometry of Riemann, once more with a start from Stokes. For its brevity, this is a remarkably good account; but it would be clearer if the author did not fear the words covariant and contravariant, and avoid all explanation of the distinction between subscript and superscript. The fourth note, entitled *The Geometry of Light*, deals condescendingly with restricted relativity.

Probably the simultaneous publication of several works on non-euclidean geometry is due, in part, to the influence of recent physical theories. The three books here reviewed are well fitted to lead to keen interest in that subject for its own sake.

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