EISENHART'S COLLOQUIUM LECTURES

Non-Riemannian Geometry. By Luther Pfahler Eisenhart. New York, American Mathematical Society, 1927. viii+184 pp.

Generalization having become a familiar process to mathematicians, it is not strange that Riemannian geometry was hardly sixty-four years old before the invention of the non-Riemannian. The former had its origin in Riemann's Habilitationsschrift (1854); the latter, in an article by Weyl (Mathematische Zeitschrift, vol. 2, 1918). Just a decade after the creation of non-Riemannian geometry Professor Eisenhart has given in his colloquium lectures an account of the essential features of the new theory with special reference to the lines along which it has been developed in this country, notably at Princeton.

The fundamental idea of non-Riemannian geometry is the association with a given manifold of a connection which forms a basis for comparing vectors at different points. In the first chapter of Eisenhart's book the asymmetric type of connection is studied, and parallelism of vectors is discussed. The author is careful to distinguish between vectors arising by parallel displacement and parallel vectors: given a vector v at a point x and a curve C passing through x, there is at any point x' of C a unique vector v' which arises by parallel displacement of v along C, whereas all vectors $\phi\cdot v'$, where ϕ is an arbitrary factor and v' is the vector just defined, are called parallel to v with respect to C. The definition of parallel displacement, which is given only for a curve, might well have been stated in the following more general form: the vector v undergoes parallel displacement as x describes any integral variety of the differential equations

$$v^i{}_{,j}\, dx^j = 0,$$

where $v^i{}_{,j}$ denotes the covariant derivative of v. If the vector field is assigned a priori so that $|v^i{}_{,j}| \neq 0$, the integral variety referred to in the suggested modification of the definition is a point, and the vector actually undergoes no displacement. If the connection is euclidean and the coordinates cartesian, the integral variety is the whole of space for any vector field with constant components. From the notion of parallelism are obtained the equations of the paths of the space, a path being defined as a curve whose tangents are parallel with respect to the curve. The paths are thus the straightest lines of the space.

In §8 incompletely integrable systems of total differential equations are discussed. The author proves a theorem which is fundamental in the subsequent treatment (Chapters II and III) of the equivalence problems. The later repetition of this theorem adapted to the particular problem in hand becomes rather monotonous. It would perhaps be sufficient to set up the integrability conditions in each case and to make a concise reference to the original statement of the theorem.

The remainder of Chapter I is devoted to the parallel displacement of a vector around an infinitesimal circuit, orthogonality of vectors, the generalization of the coefficients of rotation of an orthogonal ennuple, and the changes of connection which preserve parallelism. Of these topics, the last mentioned is perhaps the most important, since it leads to the projective theory of Chapter III.

Most of the results obtained for the asymmetric case to date are such direct generalizations of the symmetric or of the Riemannian case that their acquisition is not a great achievement. It is well, however, to develop the theory as far as possible without making the assumption of symmetry.

Chapter II deals with symmetric connections. Affine normal coordinates are defined in the usual way by means of the equations of the paths, but aside from an application to the definition of normal tensors and the process of extension, no subsequent use is made of them. If one were to judge from the way in which investigators fail to employ normal coordinates as a tool, one would conclude that they are beautiful but useless. In fact, little knowledge has been acquired so far by their use which cannot be as easily obtained with geodesic coordinates. There is an error in formula (22.8). For the correction, see Annals of Mathematics, (2), vol. 28, p. 556.

The theorem of Fermi to the effect that the Christoffel symbols are zero along any given curve in a properly chosen coordinate system is extended to symmetric connections. The proof is unnecessarily long. The following much simpler proof is indicated by Cartan (Mémorial des Sciences Mathématiques, No. 9, p. 18). Without loss of generality we may assume that $x^2=x^3=\cdots=x^n=0$ along the given curve. Assume for the new coordinates x' a transformation which is quadratic in $x^2, x^3, \cdots, x^n$ and which has for coefficients undetermined functions of x^1. Substitute these values of x' together with $x^2=x^3=\cdots=x^n=0$ and $\Gamma'=0$ in equations (28.1) in which the primes have been shifted:

$$(28.1)\qquad \frac{\partial^2 x'^i}{\partial x^j \partial x^k}+\Gamma'^i_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^j}\frac{\partial x'^\beta}{\partial x^k}=\Gamma^\alpha_{jk}\frac{\partial x'^i}{\partial x^\alpha}.$$

The coefficients of the zeroth and first orders are then determined by a system of ordinary differential equations to which Cauchy's existence theorem is applicable, and the coefficients of the quadratic terms are determined by finite relations. This same method can be applied to show that the coefficients of projective or conformal connection can be made zero along a curve. It is quite natural to ask, can we go still further and make the Γ's zero over a surface? The answer is in the negative. For taking the surface as $x^3=x^4=\cdots=x^n=0$, we must find a transformation to satisfy, among other conditions, the relations

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^1}=\Gamma^l_{j1}\frac{\partial x'^i}{\partial x^l},\qquad \frac{\partial^2 x'^i}{\partial x^j \partial x^2}=\Gamma^l_{j2}\frac{\partial x'^i}{\partial x^l},$$

in which $x^3, x^4, \cdots, x^n$ have been put equal to zero. Differentiating the first of these with respect to x^2 and the second with respect to x^1, and comparing, we find that B^i_{j12} must vanish for $x^3=x^4=\cdots=x^n=0$, B representing the curvature tensor. Hence the transformation does not always exist.

The problem of Fermi and related problems, such as that of local coordinate systems, can be regarded as modified forms of the equivalence problem in which the equivalence is not to hold throughout an n-dimensional region, but only in a sub-space defined by certain relations among the independent variables. The equivalence theorem proper, which gives a necessary and sufficient condition that equations (28.1) have a solution $x'=f(x)$ satisfying them identically in all the independent variables, is discussed in detail in §28. The proof given is for symmetric connections, but the extension to the Riemannian and asymmetric cases is clearly indicated.

Chapter III concerns the projective geometry of symmetric connections. In the reviewer's opinion the theory is not presented in its most symmetrical or attractive form because the author does not free himself from the affine connection. The whole subject can be developed as the theory of a projective invariant, the projective connection. When it is so developed, the conditions obtained are in obviously projective form, involving only the Π's and their derivatives. Formulas (34.3), for example, appear on the surface to express affine conditions because they involve the affine connection, but actually they are projective. The condition obtained in the projective fashion would be the expression defined by (35.17) equated to zero. Even equations (34.8), when needed, can be immediately deduced from (35.12) by an appeal to the notion of normal affine connection (p. 105).

The final portion of the chapter is given to the discussion of collineations in affine and projective spaces. By a collineation is meant a transformation which makes the components of the connection exactly the same functions of the new variables as of the old. The analogue for a Riemannian space is a motion or an automorphism of the quadratic form. As the author remarks, the problem of determining whether a given connection admits collineations is a particular case of the equivalence problem, and can be treated as such. The author chooses, however, to use the method of infinitesimal transformations. Except for the flat case and the rather obvious theorem of page 128, the results are not definitive. They are conditional upon the existence of solutions of certain differential equations. The use of Lie's method might, therefore, seem to have no advantages over a consideration of the system of total differential equations which express the law of transformation of the connection. On the contrary, the advantages of the method employed are two: first, the equations of condition are rendered linear; second, variety of method is secured by the introduction of continuous groups. For $n=2$ a classification of all spaces admitting continuous groups of collineations might be possible and interesting.

Chapter IV develops the geometry first of hypersurfaces and then of the general sub-spaces. The generalized equations of Gauss and Codazzi are developed for both cases. The most noteworthy result of this chapter is perhaps that cited by the author in his preface: "For a sub-space of a Riemannian space there is in general an induced metric and consequently an induced law of parallelism. There is not a unique induced affine connection in a sub-space of an affinely connected space."

The book ends with a bibliography of the books and articles to which reference is made in the text. Eisenhart's previous books have been conspicuous, among other things, for their carefully prepared and useful indexes. The omission of such an index from the present work is to be regretted. With the growth of mathematical literature, it is becoming more and more inconvenient to be forced to leaf through a whole book to locate a desired bit of information.

Those wishing an introduction to the subject will find *Non-Riemannian Geometry* a useful book. A familiarity with the methods of tensor analysis is, of course, presupposed on the part of the reader. The subject matter is carefully presented and the manipulations are easy to follow. The theorems are clearly stated and are proved in a straightforward manner, a number of the proofs being original with the author. The chief defect in the book—and this seems serious in a set of colloquium lectures—is the absence of any attempt to comment upon the significance of the results or to point out possible lines for future development.

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THREE BOOKS ON DIFFERENTIAL EQUATIONS

An Elementary Treatise on Differential Equations and their Applications. New edition, revised and enlarged. By H. T. H. Piaggio. London, G. Bell and Sons, 1928. 18+256+27 pp.

Gewöhnliche Differentialgleichungen. By J. Horn. Zweite, völlig umgearbeitete Auflage. (Göschens Lehrbücherei.) Berlin, Walter de Gruyter, 1927. 8+197 pp.

Ordinary Differential Equations. By E. L. Ince. London and New York, Longmans, Green, 1927. 8+558 pp.

Professor Piaggio's *Differential Equations* was first published in May, 1920, and was reprinted four times during the next six years. "The object of this book is to give an account of the central parts of the subject in as simple a form as possible, suitable for those with no previous knowledge of it, and yet at the same time to point out the different directions in which it may be developed." The only previous knowledge assumed is that of the differential and integral calculus. The style is admirably adapted to a text for beginners and the large number of examples with answers furnishes adequate drill material.

The usual standard forms of ordinary and partial differential equations occupy the greater part of the book: one chapter is devoted to numerical approximations and another to existence theorems. In the revised edition more examples have been included and a new chapter "Miscellaneous Methods" has been added, dealing with a number of disconnected topics, which may be regarded as in the nature of supplementary reading. The author calls particular attention to some difficulties in the theory of singular solutions, for which he is indebted to some unpublished work by Mr. H. B. Mitchell, formerly Professor at Columbia University.