GERMAN EDITION OF DICKSON ON ALGEBRAS

Algebren und ihre Zahlentheorie. By L. E. Dickson. *Mit einem Kapitel über Idealtheorie von A. Speiser.* Zürich und Leipzig, Orell Füssli Verlag, 1927. 308 pp.

Another notable advance in the development of the subject of linear algebras occurred with the appearance of this book. The mathematical world naturally expects a high standard in the case of treatises by Professor Dickson and the present one will not be found disappointing. While to a certain extent a revision and translation into the German language of his earlier book, *Algebras and their Arithmetics* (reviewed in this Bulletin, vol. 30 (1924), pp. 263–270, by Professor Olive C. Hazlett), it contains much material that is new and interesting, particularly along the lines of extension of the theory of division algebras and of the theory of numbers. Included in the latter category is the final chapter on ideal theory by Professor A. Speiser, under whose supervision the translation was made and who has written the preface.

On the basis of the new results contained in this book the first award of the Cole Prize by this Society was made at its meeting in Chicago in April, 1928. It is worth noting also that some years earlier the annual prize of the A. A. A. S. was awarded to Professor Dickson for a paper of unusual merit on the same subject.

It will be sufficient here to discuss the chief ways in which this book differs from the earlier one. Aside from a rearrangement of the material the first thing that strikes the reader's attention is the extension of the theory of linear associative algebras contained in sections 34–44. While based on an article by the author in volume 28 of the Transactions of this Society, it nevertheless differs considerably from that in that it avoids the use of the theory of finite groups. Whether this is really an advantage may perhaps be debatable.

The author deals here with a division algebra A over a general field K that contains a principal unit and that has the four properties: (I) A is of order n^2; (II) it contains an element i that satisfies an equation, $f(\omega)=0$, irreducible in K; (III) the only elements in A that are commutative with i are polynomials in i with coefficients in K; (IV) all the roots of $f(\omega)=0$ are rational functions of i with coefficients in K. In view of a theorem proved later, however, the only condition, as the author points out, that really restricts in any way the generality of the attack on the determination of associative division algebras is the last.

For the case where the irreducible equation $f(\omega)=0$ is "cyclic," to use the terminology of finite groups, algebras of this sort had been determined previously by the author and had been discussed in the English edition. The notable advance here consists in the use of non-cyclic equations. The chief difficulty of the problem lies in the imposition of the associative law, which leads to some conditions that are not particularly simple. Any doubt that may arise as to the existence of algebras of this type is dispelled however by detailed consideration of special cases that has been carried out. A recent contribution along this line is a Chicago dissertation by A. A. Albert, in which the division algebras of order 16 are determined.

In a later chapter the author discusses generalized quaternion algebras over the field of rational numbers, these being defined by elements i, j that satisfy the conditions $i^2=-1$, $j^2=\tau$, $ji=-ij$, where τ is restricted to certain rational integral values. As in the case $\tau=-1$, which has been treated at considerable length by A. Hurwitz in his book *Vorlesungen über die Zahlentheorie der Quaternionen*, he develops a theory of integral elements of such algebras that parallels closely the corresponding theory for quadratic algebraic fields, allowance being made for the failure of the commutative law. For certain values of τ the analog of the euclidean algorithm for rational integers is shown to hold and a theory of representation of integers by certain quaternary quadratic forms is obtained. The results established here have very likely furnished the inspiration for the recent work by the author and some of his students on the representation of integers by more general forms of this sort. A result of particular interest concerns the solution of diophantine equations of the type $f=y_1y_2\cdots y_k$, where f represents certain quaternary quadratic forms.

The following chapter on the general number theory of algebras has been to a large extent rewritten and supplemented by new and interesting material. This theory, restricted in the English edition to algebras over a rational field, is here extended to algebras over a general algebraic field, being based on independent work in this direction by the author and by Professor Olive C. Hazlett. "Integral elements" of an algebra are here defined as those that satisfy equations with coefficients that are integers in the field over which the algebra is defined and with leading coefficient unity, and that belong to a "maximal integral domain" of the algebra, that is, a set of elements that reproduce themselves under addition, subtraction, and multiplication, and that are not included in a larger domain of the same sort. The second condition is not a consequence of the first, as is the case with algebraic integers, and hence all integral elements of an algebra may not be included in one integral domain. The definition given here seems simpler than that in the English edition where the "rank equation" was employed. As a justification for it there is proved the important theorem that integral elements exist in every algebra.

As in the earlier edition, the theory of integral elements for a general algebra is shown to be associated with that for a semi-simple subalgebra, and the latter to be dependent on the corresponding theory for simple algebras. This in turn leads to the consideration of matrices whose elements are integral quantities of a division algebra. In particular, some interesting theorems are obtained for the case where this division algebra possesses a euclidean algorithm. The theory of the factorization of such matrices leads to a remarkable result concerning the solution, in integers belonging to algebraic fields that possess this property, of an equation of the form $d=y_1y_2\cdots y_k$, where d represents a determinant whose n^2 elements are independent variables. An application is given to the solution of the equation $x^2+y^2+z^2=u^2+v^2+w^2$ in integers belonging to such a field. It is perhaps worth noting that the values for the six indeterminates obtained by the method might have to be multiplied by 2 (or possibly some factor of 2) in order that they should become integers.

Other new results are a theorem concerning maximal integral domains in a semi-simple algebra, a determination of the bases of all maximal domains of integral quaternions in a quadratic field, a proof that every normal division algebra of order n^2 is of rank n, and a generalization of Cayley's non-associative algebra of order 8. Further additions are a determination of all division algebras of rank 2 and those of order $\leqq 4$, certain theorems concerning algebraic numbers, a discussion of "quasi-fields," including a proof that every finite quasi-field is a field, and theorems concerning the extension of fields by the adjunction of indeterminates and concerning algebras in quasi-fields.

The final chapter on the theory of ideals in rational algebras, which was contributed by A. Speiser, seems interesting and well deserving of inclusion in spite of the fact that the introduction of ideals fails to restore unique factorization in contrast to the situation in algebraic fields. It is necessary here to distinguish between "right-sided," "left-sided," and "two-sided" ideals. The discussion is devoted in considerable part to the set of residues with respect to a two-sided ideal modulus. As these constitute an algebra over a finite field, a number of the fundamental theorems proved in preceding chapters apply to them. In the case of a two-sided prime ideal it is shown that the residues form a system that is isomorphic with respect to addition and multiplication with the totality of all matrices of a particular order with elements in a particular (finite) Galois field. An important theorem on algebraic numbers is generalized by proving that the prime divisors of the discriminant and only these are divisible by the squares of (two-sided) prime ideals.

The book is remarkably free from obscurities or inaccuracies. A few places of this sort in the English edition that the reviewer had noted have in nearly every instance been removed in the revision. An exception is possibly the proof of the theorem (section 71 of the new edition) that, if an algebra contains an idempotent element, it contains at least one such that is primitive. Here it appears to be assumed that the relation, $euAue < eAe$, follows as a consequence from $uAu < A$, a step that does not seem obvious to the reviewer. Only a slight modification of the proof however is necessary to remove this (apparent) obscurity.

In the proof of Theorem 8 in section 81 it may not be evident to a reader what is the relation between $[B]$ and B. In view of the proof on the next page that $A-N=(B-N_1)\times M$, where $N_1=N\wedge B$ is invariant in B, it follows that $[B]=B-N_1$.

The book is very attractive in its make-up, and the translation and proof-reading have been very carefully done. All concerned are to be congratulated on the result.

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