THE THANKSGIVING MEETING IN CINCINNATI

The two hundred sixty-fourth, thirtieth regular Western, meeting of the Society was held at the University of Cincinnati on Friday and Saturday, November 30–December 1, 1928. Over one hundred persons attended the meeting, among whom were the following fifty members of the Society:

B. E. Allen, W. E. Anderson, C. L. Arnold, Bareis, Barnett, Birkhoff, Blumberg, Brand, Byrne, Coble, L. W. Cohen, H. T. Davis, P. D. Edwards, Feinler, Fry, C. A. Garabedian, Hancock, E. R. Hedrick, Archibald Henderson, Ingraham, M. M. Johnson, Kersten, Kindle, Kuhn, Latimer, LeStourgeon, MacDuffee, Metz, Michal, C. N. Moore, Mullings, T. S. Peterson, Pollard, Rainich, Rasor, C. E. Rhodes, Roth, Shohat, Sibert, W. G. Simon, Spenceley, Tripp, J. H. Weaver, Wiley, K. P. Williams, C. O. Williamson, Winston, Yanney, Yowell, Zehring.

Friday afternoon was devoted to addresses given upon invitation of the Program Committee. Professor E. R. Hedrick spoke on *Recent developments regarding non-analytic functions*. Professor Archibald Henderson spoke on *Relativity: Survey and Outlook*. These papers were followed by a lecture on *The mathematical basis of art* by Professor G. D. Birkhoff. The paper of Dr. T. C. Fry on *The use of continued fractions in the design of electrical networks* which appeared on the program on Saturday morning was also upon invitation of the Program Committee.

On Friday evening fifty-five members and their guests attended dinner in the Hotel Gibson. Professor Hancock acted as toastmaster. The toastmaster called upon Professor Birkhoff to speak of his recent trip around the world, on Dr. Fry to talk of mathematics in industry, while he himself spoke on the future of mathematical meetings in the Ohio Valley region, and called on Professors Hedrick, C. N. Moore and Blumberg to discuss the same topic. The dinner was a complimentary one for the out of town members of the Society.

On Saturday morning a resolution was passed thanking the authorities of the University of Cincinnati, the Chamber of Commerce of the City of Cincinnati, the Local Press, the Hotel Gibson and the Local Committee, of which Professor C. N. Moore was Chairman, for the excellent arrangements that had been made for the program and for the comfort and pleasure of the members of the Society. A resolution was also passed thanking Professor Hedrick, Professor Henderson, Professor Birkhoff, and Dr. Fry for their interesting and stimulating addresses.

The papers whose abstracts appear below were read as follows: papers 1–7 on Friday morning, and papers 8–17 on Saturday morning. Papers 14, 15 and 17 were read by title, the rest in person. Professor E. R. Hedrick presided Friday morning, Professor Coble presided during the addresses of Professors Hedrick and Henderson; Professor C. N. Moore presided during Professor Birkhoff's lecture; and Professors Kuhn and Birkhoff presided Saturday morning. Mr. Miller was introduced by Professor R. L. Wilder, and Mr. Churchill by Professor Rainich.

1. Professor J. A. Shohat: *On the polynomial and trigonometric approximation of measurable bounded functions, on a finite interval.*

This paper shows that the notion of the polynomial (or trigonometric) "best approximation" (in Tchebycheff sense) can be extended, *in two ways*, to the case of measurable bounded functions defined on a finite interval, making use: (a) of the upper bound, or (b) the "measurable upper bound" (C. Haskins) of such functions. The second kind of "polynomials of best approximation" is more important, for it gives an indefinitely close approximation, as in the case of continuous functions. It is also closely related to $\min\int_a^b p(x)\,|f(x)-G_n(x)|^m dx$, as $m\rightarrow\infty$ ($p(x)\geqq 0$, $f(x)$ are defined on (a, b), $G_n(x)$ is an arbitrary polynomial of degree $\leqq n$). Considering the same minimum for m, $n\rightarrow\infty$, we get a sequence of polynomials converging, under very general conditions, to $f(x)$ uniformly over the whole interval (a, b), with an approximation of the same order (for $n\rightarrow\infty$), as the best approximation. (See the work of G. Pólya, D. Jackson, and the author.)

2. Dr. L. W. Cohen: *On sequences of functions, of sets, and of Lebesgue integrals.*

For $f(x)\geqq 0$, we define $h(f; A)$; $H(f; A)$ as the set of all points $P(x, y)$ such that $x\subset A$, $0\leqq y<f(x)$; $0\leqq y\leqq f(x)$ respectively. Theorem I: If $f_n\geqq 0$

on A and $\lim_n f_n = f$, then (1) $h(f; a) \subset \lim\inf_n h(f_n; A) \subset \lim\inf_n H(f_n; A) \subset H(f; A)$; (2) $h(f; A) \subset \lim\sup_n h(f_n; A) \subset \lim\sup_n H(f_n; A) \subset H(f; A)$, and conversely. Let W_p be the cube $-p \leqq x_i \leqq p (i = 1, 2, \cdots, n; p = 1, 2, \cdots)$ and A_k be a set in n-space. Theorem II: If meas $A_k < +\infty$ and $\lim_p$meas $(A_k - A_k W_p) = 0$, uniformly in k, then $\lim\sup_k$ meas $A_k \leqq$ meas $\lim\sup_k A_k$. Theorem III: If $\lim_k$meas $A_k =$ meas A and meas $\lim\inf_k A_k =$ meas A then $\lim_p$meas $(A_k - A_k W_p) = 0$, uniformly in k. From I and II it follows that the usual sufficient condition $|f_n| \leqq S$, where S is summable, can be replaced by the more general condition $\lim_{\tau = +\infty} \int_A \max(0; |f_n| - \tau) = 0$, uniformly in n, in theorems on the term-by-term integration of sequences of functions. From I and III it follows that the above condition is necessary in the case of term-by-term integration of monotonic sequences or of sequences of non-negative functions.

3. Professor H. T. Davis: *A general theory of operators with explicit application to differential equations of infinite order with polynomial coefficients of degree p.*

The present theory of operators is founded upon the Pincherle-Bourlet generatrix equation $[X \cdot F] = 1$ where the bracket symbolizes the product of the differential operators $X(x, z)$ and $F(x, z)$ in which $z = d/dx$. The case where $F(x, z)$ expands into a power series in z with constant coefficients leads to the methods of the Heaviside calculus and to formal solutions of difference equations with constant coefficients. When $F(x, z)$ expands into a series in negative powers of z the operational solution of the Volterra integral equation of the closed cycle is obtained. When the coefficients of $F(x, z)$ are polynomials of degree not greater than p the solution of the generatrix equation yields three formal series: (1) in powers of z; (2) in negative powers of x; (3) in inverse factorials in x. The regions of convergence of the solutions obtained from these formal operators are discussed.

4. Professor C. C. MacDuffee: *The discriminant matrices of a linear associative algebra.*

A sufficient condition in order that the first matrices $R_1, R_2, \cdots, R_n$ of the basal numbers of an algebra be linearly independent is found to be the non-singularity of a certain matrix called the *first discriminant matrix*, since its determinant proves to be the discriminant of the algebra as defined by Speiser. From the second matrices $S_1, S_2, \cdots, S_n$ of the basal numbers we obtain similarly a *second discriminant matrix*. These matrices are symmetric, and under a linear transformation of units are transformed like the matrix of a quadratic form. Other properties are noted.

5. Professor W. E. Roth: *On the solution of the matric equation $P(A, X) = 0$ for X commutative with A.*

The present paper takes up the matric equation $P(A, X) = 0$, where $P(\lambda, \mu)$ is a polynomial in λ and μ, and where A is a given square matrix. Solutions of the equation subject to the restriction that they be commutative with A are found. This restriction is much less severe than that imposed in a paper previously announced by the writer, where solutions of the same equation expressible as polynomials in the given matrix were sought. See abstract in this Bulletin, vol. 34, p. 151.

6. Mr. E. W. Miller: *The most general closed point set in n-dimensional space through which it is possible to pass an arc.*

It was shown by R. L. Moore and J. R. Kline in 1919 (Annals of Mathematics, vol. 20, pp. 218–223) that in order that a closed and bounded *plane* point set M should be a subset of an arc, it is necessary and sufficient that every component of M should be either a point or an arc, and that if t is an arc of M, no point of t, with the exception of its end points, is a limit point of $M-t$. It is shown in the present paper that if M is a closed and bounded point set in n-dimensional euclidean space, these same conditions are necessary and sufficient in order that M should be a subset of an arc.

7. Professor Henry Blumberg: *Extension and applications of the theorem on the measurable bounds of an arbitrary function.*

This theorem, communicated at the New York 1928 April meeting, is as follows: If $f(x)$—to confine the statement to the case of one variable—is an arbitrary real function, there exist two measurable functions $m_1(x)$ and $m_2(x)$ such that $m_1(x) \leqq f(x) \leqq m_2(x)$ almost everywhere, and the "metric approach," in the sense of exterior measure, via the curve $y=f(x)$ is non-infinitesimal at every point of $y=m_1(x)$ and $y=m_2(x)$. This theorem seems to be of marked utility in yielding extensions of certain properties of measurable functions to unconditioned functions. Thus the author shows that the Denjoy-Young theorems on derivatives of measurable functions carry over rather directly to general functions; likewise, for example—in a certain sense—the Arzela-Fréchet necessary and sufficient condition that an infinity of measurable functions be such as to allow the extraction of a sequence, convergent on the average. There is application, too, in the theory of integration. The theorem is extended to give information also concerning the manner of the clustering of the points of $y=f(x)$ between its measurable bounds.

8. Professor Marie M. Johnson: *Tensors of the calculus of variations.*

The tensors, considered in this paper, are connected with the non-parametric problem of the calculus of variations for a space of $n+1$ dimensions. The n expressions in the Euler differential equations, multiplied by dx, and an additional function form a covariant tensor of rank 1. An application is made of this tensor to "canonical" transformations as defined by Carathéodory. It is found that the expression which, when set equal to zero, defines transversality is an invariant. The quadratic form of the Legendre condition after it is multiplied by $1/dx$ becomes an invariant. When the Weierstrass E-function is transformed, a factor is introduced. There are $n+1$ functions which are connected with the Jacobi differential equations that form a covariant tensor. The laws of transformation of the two determinants which are used to find the conjugate points of the Jacobi condition are discussed. In connection with fields of extremals it is shown that the expression in the Hamilton-Jacobi partial differential equation is an invariant and that the necessary conditions for an integral which is independent of the path have tensor character.

9. Professor Louis Brand: *The calculus of variations from a vector standpoint.*

This paper will appear in full in an early issue of this Bulletin.

10. Dr. T. C. Fry: *The use of continued fractions in the design of electrical networks.*

The problem in electrical circuit theory which is most frequently touched upon in textbooks is "Given a particular electrical system and a known driving force to find how the system responds." There is, however, a converse problem which is equally important, viz., "Given a known driving force and the response which it is desired to produce, to find a system which will so respond." One method of attacking this problem is through the medium of continued fractions, particularly those of the Stieltjes type. The present paper explains in a general way the part that continued fractions play in the solution of such problems and discusses some extensions which must be made in the Stieltjes theory in order to meet certain mathematical restrictions that are inherent in the technical problem.

11. Professor G. D. Birkhoff: *A theorem on the singular points of ordinary differential equations.*

In the differential equations $dx_i/dt=X_i$, $(i=1, 2)$, in which the real analytic functions X_i vanish at the singular point (0, 0), the characteristic numbers m_1, m_2 may be either (1) real and of the same sign, (2) real and of opposite signs, or (3) conjugate imaginaries. Certain associated series converge in general cases (1) and (3), but may diverge in case (2). Nevertheless this paper establishes in this case for m_1/m_2 irrational, that by a one-to-one local transformation of the x_1, x_2 plane, in which the functions involved are continuous together with all their partial derivatives, analytic except along the two invariant analytic curves through (0, 0) and possessed of certain other properties, the differential equations reduce to $dx_i/dt=m_ix_i$, $(i=1, 2)$. This is essentially the same normal form previously obtained in the other two general cases by means of a one-to-one analytic transformation based upon the convergent series available in these cases.

12. Professor G. Y. Rainich: *Plane waves in curved space-time.*

The group of Lorentz transformations is found to which the electromagnetic field of a plane wave belongs, and a metric is sought which admits an isomorphic group. Using curvilinear coordinates introduced before, the general symmetric tensor of rank two admitting the group is found: it is taken as the fundamental tensor of a metric and subjected to the condition that the contracted Riemann tensor of that metric has the form of an electromagnetic stress energy tensor. The consequence of this restriction is that the space has zero curvature. This result is interpreted as indicating that in general relativity plane waves do not exist—a conclusion arrived at by Baldwin and Jeffery in a different way. It follows that the only type of fields in curved space-time which possess symmetry properties of radiation are fields with plane singularities. It is indicated that a field with a great number of such singularities may be considered as having the symmetry of a plane wave if local discontinuities are disregarded much as a part of space with many point singularities corresponding to material particles may be regarded as a continuous body by neglecting local discontinuities.

13. Mr. R. V. Churchill: *On the geometry of the Riemann tensor.*

At any point of a four-dimensional Riemann space a tensor of the second type is determined by a skeleton (that is, a set of four mutually perpendicular directions intrinsically related to the tensor) and three numbers. Moreover the skeleton is uniquely determined by this tensor. A tensor of the first type is determined by a skeleton and five numbers, but the skeleton is not uniquely determined by this tensor. From any one skeleton and its set of five numbers five other skeletons with their corresponding sets of five numbers can be found. Each of these six skeletons with its set of numbers serves in the same way in the determination of their tensor. It has been shown that the Riemann tensor at any point is the sum of a tensor of the first type and a tensor of the second type. Hence the Riemann tensor at a point of a four-dimensional Riemann space is determined by two skeletons and eight numbers. In this paper most of these results are obtained by using six-vectors for the arguments of the Riemann tensor after a restatement and extension of the theory of six-vectors is made.

14. Dr. I. M. Sheffer: *On the theory of sum-orthogonal polynomials.*

Let there be given two sets of real numbers: $\{p_k\}$, $\{x_k\}$, $k=0, 1, \cdots, \infty$, with $p_k>0$ and $x_i\neq x_j$ for $i\neq j$. We denote by $S[u(x)]$ (when it exists) the weighted sum for $u(x)$ relative to $\{p_k\}$ over the range $\{x_k\}$: $S[u(x)] =\sum_{k=0}^{\infty} P_k u(x_k)$. Two functions $u(x)$, $v(x)$ are defined to be sum-orthogonal if $S[uv]=0$. This paper is concerned with the existence and properties of a (unique) system of polynomials $\{T_n(x)\}$ which are normal and sum-orthogonal.

15. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem and continuous curves in n-space.*

The author proves that a continuous curve (bounded) in n-space without a cut point which satisfies the Janiszewski-Mullikin theorem must be the simple sphere of analysis situs, establishing in a sense a "converse theorem." Using generalizations of certain plane theorems he is able to discard the condition of no cut point and to discuss in full the most general solution.

16. Professor K. P. Williams: *A generalization of the Cauchy-Riemann equations.*

The sum, product, and quotient of two analytic functions are analytic. This shows that there is a formal connection between the rules for differentiating the product and quotient of two functions and the Cauchy-Riemann equations, and suggests the following problem considered in this paper. Let $u(x, y)$, $v(x, y)$ and $u^*(x, y)$, $v^*(x, y)$ be two solutions of $au_x+bu_y+cv_x+dv_y=0$ where a, b, c, d are constants. Let it be required to annex other conditions such that U and V will also satisfy the given and annexed conditions, where $U+iV$ is the sum, product, or quotient of $u+iv$ and u^*+iv^*. It is found that one is led to the system of equations $Au_x+u_y+v_x=0$, $u_x-Av_x-v_y=0$. These equations can be regarded as a natural generalization of the Cauchy-Riemann equations. It is found that $u+iv$ can be considered as an analytic function of $(x+iv)-Ay$.

17. Professor A. D. Campbell: *Note on the Plücker equations for plane algebraic curves in the Galois fields.*

This paper appears in full in the November-December 1928 issue of this Bulletin.

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Associate Secretary