ZEROS OF A FUNCTION AND OF ITS DERIVATIVE*

BY W. J. TRJITZINSKY

Macdonald has proved the following theorem.† Let $f(z)=u(x, y)+iv(x, y)$ be a function of z analytic throughout the interior of a single closed curve C, defined by the equation $|f(z)|=M$, where M is a constant. Let C be an ordinary curve in the sense that if ψ be the angle which the tangent to C makes with the x-axis, and if the point of contact of the tangent describes the curve C, ψ will increase by 2π. Then the number of zeros of $f(z)$ in this region exceeds the number of zeros of the derivative $f\ (z)$ by unity. Our purpose is to generalize this result by showing the conclusion holds true even when M is not a constant, but a function $M=M(x, y)$ analytic and greater than zero within sufficiently large intervals, and C is a curve single, closed, and ordinary (in the sense mentioned above), and satisfying the equation $|f(z)|=M(x, y)$, provided that the point $(x=x(\theta), y=y(\theta))$ describes C once when θ changes from 0 to 2π; where $x(\theta)$ and $y(\theta)$ are periodic functions satisfying the equations

$$(1)\qquad \frac{u(x,y)}{M(x,y)} = \cos\theta, \qquad \frac{v(x,y)}{M(x,y)} = \sin\theta.$$

The solvability of these equations implies that the Jacobian of the left members does not vanish identically.

On C, let $f(z)=M(x, y)e^{i\theta}$. Then there results a pair of equations (1) whose solutions are $x_i=x_i(\theta)$ and $y_i=y_i(\theta)$ of period 2π in θ. By hypothesis, among them there is at least one solution, say $x=x(\theta)$, $y=y(\theta)$, representing the curve C, and the point $(x=x(\theta), y=y(\theta))$ describes C once as θ varies from 0 to 2π.

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† Proceedings of the London Society, vol. 29 (1898), pp. 576–577; Proceedings of the London Society, (2), vol. 15 (1916), pp. 227–242.

Letting $M(x(\theta), y(\theta)) = N(\theta)$, we observe that $N(\theta)$ as well as $N'(\theta)$ is of period 2π, and for all θ, $N(\theta) > 0$, so that when z describes C and θ varies from 0 to 2π, the complex quantity $N'(\theta) + iN(\theta)$ describes a closed curve entirely above the real axis; its modulus will return to the initial value and the variation of its argument will be zero. Now on C we have

$$f(z) = N(\theta) \cdot e^{i\theta},$$

$$f'(z) = e^{i\theta} \cdot \frac{d\theta}{dz} \cdot [N'(\theta) + iN(\theta)],$$

$$f''(z) = e^{i\theta} \cdot f\left\{ [N'(\theta) + iN(\theta)] \cdot \left[\frac{d^2\theta}{dz^2} + i\left(\frac{d\theta}{dz}\right)^2\right] + [N''(\theta) + iN'(\theta)]\left(\frac{d\theta}{dz}\right)^2 \right\}.$$

The excess e of the number of zeros of $f(z)$ over the number of zeros of $f'(z)$ within C is

$$e = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}\int_C \frac{f''(z)}{f'(z)}dz$$

$$= -\frac{1}{2\pi i}\int_C \frac{\left(\frac{d^2\theta}{dz^2}\right)}{\left(\frac{d\theta}{dz}\right)}dz - \frac{1}{2\pi i}\int_C \frac{d\theta}{dz}\left[\frac{N''(\theta) + N'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]dz.$$

The quantity

$$-\frac{1}{2\pi i}\int_C \frac{\left(\frac{d^2\theta}{dz^2}\right)}{\left(\frac{d\theta}{dz}\right)} \cdot dz = 1,$$

as can be seen from Whittaker's* presentation of the theorem cited. Hence we have

* Whittaker and Watson, *Modern Analysis*, 3d ed., p. 121.

$$e = 1 - \frac{1}{2\pi i}\int_C \left(\frac{d\theta}{dz}\right)\cdot\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]dz$$

$$= 1 - \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]d\theta,$$

since θ varies from 0 to 2π when z describes C. Moreover

$$e = 1 - \frac{1}{2\pi i}\cdot\left\{\log\left[N'(\theta) + iN(\theta)\right]\right\}_0^{2\pi} + \frac{1}{2\pi i}\left[\log N(\theta)\right]_0^{2\pi}.$$

We know that $N(\theta)$ is real; hence $[\log N(\theta)]_0^{2\pi}=0$. On the other hand, the variation of the argument of $[N'(\theta)+iN(\theta)]$ as θ changes from 0 to 2π is zero, so that $\log [(N'(\theta)+iN(\theta))]_0^{2\pi}=0$. Hence $e=1$. This proves the theorem.

THE UNIVERSITY OF TEXAS

LINEAR INEQUALITIES IN GENERAL ANALYSIS*

BY L. L. DINES

1. *Introduction.* In his studies in general analysis, E. H. Moore has developed† a theory of the linear functional equation

$$\xi + J\kappa\xi = \eta.$$

Here ξ and η denote functions (the latter given, the former to be determined) belonging to a class $\mathfrak{M}$ of real-valued functions on a general range $\mathfrak{P}$. The kernel function κ belongs to a class $\mathfrak{K}$ which is well defined in terms of the fundamental class $\mathfrak{M}$. A sufficient foundation for the theory is laid by means of postulates upon the class $\mathfrak{M}$ and the functional operation J.

The purpose of the present paper is to consider the linear *inequality*

$$(1) \qquad \xi + J\kappa\xi > 0,$$

* Presented to the Society, September 8, 1927.

† *On the foundations of the theory of linear integral equations*, this Bulletin, vol. 18, pp. 334–362.