THE DUAL OF A LOGICAL EXPRESSION*

BY B. A. BERNSTEIN

The Peirce-Schröder law of duality in Boolean logic is that for every proposition in the logic there exists another proposition, obtained from the former by interchanging the operations $+$, $\times$ and the elements 0, 1. The rule for obtaining the dual of a logical expression involved in this law is convenient enough when no special form is desired for the dual, but it is generally not at all convenient if the dual is required in the very much desired *normal* form. The main object of this note is to obtain a convenient rule for writing down the dual of an expression in the normal form.

Let $a, b, \cdots, l$ be the *discriminants* of a logical function $f(x, y, \cdots, t)$. Then f, developed normally with respect to its arguments, is given by

$$(1)\quad f(x,y,\cdots,t) = axy\cdots t + bxy\cdots t' + \cdots + lx'y'\cdots t',$$

where the primes indicate negation. If f_1 denote the dual of f, we have by the rule of Peirce and Schröder

$$f_1 = (a_1 + x + y + \cdots + t)(b_1 + x + y + \cdots + t')\cdots (l_1 + x' + y' + \cdots + t'),$$

where $a_1, b_1, \cdots, l_1$ are the respective duals of $a, b, \cdots, l$.† Or, developed normally with respect to $x, y, \cdots, t$,

$$(2)\qquad f_1 = l_1xy\cdots t + \cdots + b_1x'y'\cdots t + a_1x'y'\cdots t'.$$

Let us call two discriminants of a function *conjugate* when the arguments associated with one are the respective negatives of those associated with the other. Then (2) tells us

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† The discriminants $a, b, \cdots, l$ may be 0, 1, or functions of other elements, $\alpha, \beta, \cdots, \mu$; so that $a_1, b_1, \cdots, l_1$ will in general be different from $a, b, \cdots, l$ respectively.

that *any discriminant of the dual of a function is the dual of the conjugate of the corresponding* discriminant of the function.* We then have the following rule.

RULE. *To obtain the dual of a logical expression, develop the expression normally with respect to any of its elements and change each discriminant to the dual of its conjugate.*

This is the desired rule. Thus, the dual of

$$axy + abx'y + x'y'$$

is

$$0 \cdot xy + (a + b)xy' + x'y + ax'y'.$$

Our rule brings to light some interesting facts about duals. We have at once that *a function is self-dual if each discriminant is the dual of its conjugate.*

Further, the negative of f given by (1) is

$$(3) \quad f' = a'xy \cdots t + b'xy \cdots t' + \cdots + l'x'y' \cdots t'.$$

So that

$$(f')_1 = (l')_1 xy \cdots t + \cdots + (b')_1 x'y' \cdots t + (a')_1 x'y' \cdots t'$$

and

$$(f_1)' = (l_1)'xy \cdots t + \cdots + (b_1)'x'y' \cdots t + (a_1)'x'y' \cdots t'.$$

But

$$(k')_1 = (k_1)' \quad \text{for} \quad k = 0, 1, \alpha + \beta, \alpha\beta.$$

Hence

$$(4) \qquad (f')_1 = (f_1)' = f_1'.$$

That is, *the dual of the negative of a function is the negative of the dual of the function.*†

With the aid of (4), we get

$$(5) \qquad (((f')_1)')_1 = (((f_1)')_1)' = \cdots = f_{11}'' = f.$$

And so, *a function is left unaltered if operated on in any order by duals and negatives each taken an even number of times.*

* By *corresponding* discriminants of two functions we mean discriminants associated with identical products of the arguments.

† Sheffer's operation $a|b$ is seen to be the dual-negative of ab.

Again, generally $f_1 \neq f'$. Under what condition will $f_1 = f'$? From (2) and (3) we see that *the dual of a function is the same as the negative of the function if each discriminant is the dual-negative of its conjugate.*

Finally, if we have a relation $f = 0$, then in general $f_1 \neq 1$, though always $f' = 1$. What is the condition that $f_1 = 1$ when $f = 0$? By means of (2) and (3) we find this condition to be *the same as the condition that* $f_1 = f'$.

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THE HEAVISIDE OPERATIONAL CALCULUS*

BY H. W. MARCH

In a number of recent papers, Carson† has made a definite advance in the study of the Heaviside operational calculus by showing that the solution of an operational equation of the type in question can be obtained from an integral equation. Having done this, he was able to discuss Heaviside's three principal rules and to derive a number of important theorems by the use of which it is possible to solve by operational methods, problems to which Heaviside's rules are not directly applicable.

Somewhat earlier Bromwich‡ and Wagner§ solved, by the use of contour integrals in the complex plane, problems to which one of Heaviside's rules is applicable. They noted that the corresponding rule of Heaviside, the expansion theorem, follows at once from a calculation of the residues at the poles of the integrand in the case of a suitably restricted

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† J. R. Carson, Bell Technical Journal, vol. 4 (1925), pp. 685–761; vol. 5 (1926), pp. 50–95, 336–384; this Bulletin, vol. 22 (1926), pp. 43–68. See also P. Levy, Bulletin des Sciences Mathêmatiques, vol. 50 (1926), pp. 174–192.

‡ T. J. I'A. Bromwich, Proceedings of the London Society, (2), vol. 15 (1916), pp. 401–448; see also H. S. Carslaw, Philosophical Magazine, vol. 39 (1920), pp. 603–611.

§ K. W. Wagner, Archiv für Elektrotechnik, vol. 4 (1916), pp. 159–193.