FUNCTIONS EXPANSIBLE IN SERIES*

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In the Transactions of this Society† Hopkins has stated and proved the following theorem.

THEOREM I. *If $f(x)$ is a function analytic in the interior and on the boundary of a circle centered at $x=0$ and of radius x_0, $0<x_0<\pi$, which involves in its power series expansion only powers of x of indices congruent to* 2 (mod 3), *and which has a continuous second derivative for real values of x in the interval $0\leqq x\leqq\pi$, then the formal development of $f(x)$ in a series whose terms are the characteristic functions of the differential system*

$$\frac{d^3u}{dx^3}+\rho^3u=0,\qquad u(0)=u'(0)=u(\pi)=0,$$

converges uniformly to $f(x)$ in the interval $0\leqq x\leqq x_0$.

Hopkins proved further that the development converges uniformly to $f(x)$ in the interior of an equilateral triangle centered at $x=0$ and having one vertex at $x=x_0$. The following theorem in which the adjoint differential system appears is obtained from Theorem I by the change of variable $x'=\pi-x$.

THEOREM II. *If $f(x)$ is a function analytic in the interior and on the boundary of a circle centered at $x=\pi$ and of radius x_1, $0<x_1<\pi$, which involves in its power series expansion only powers of $(\pi-x)$ of indices congruent to* 2 (mod 3), *and which has a continuous second derivative for real values of x in the interval $0\leqq x\leqq\pi$, then the formal development of $f(x)$ in a series*

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† J. W. Hopkins, Transactions of this Society, 1919, pp. 245, et seq. Published by D. Jackson.

whose terms are the characteristic functions of the differential system

$$\frac{d^3v}{dx^3} - \rho^3 v = 0, \qquad v(\pi) = v'(\pi) = v(0) = 0,$$

converges uniformly to $f(x)$ *in the interval* $x_1 \leqq x \leqq \pi$.

It can be shown further that the latter development converges uniformly to $f(x)$ in the interior of an equilateral triangle centered at $x=\pi$ and having one vertex at $x=x_1$.

The purpose of this note is to determine whether there are functions satisfying the conditions of both the above theorems; and if there are, to see whether the corresponding series may have a common range of convergence.

We derive first conditions which must be satisfied by a function $f(x)$ of the type demanded by both theorems. We must have $f(x)=x^2\phi(x^3)$ and $f(x)=(\pi-x)^2\Phi[(\pi-x)^3]$, where $\phi(x^3)$ and $\Phi[(\pi-x)^3]$ are analytic in x^3 at $x=0$ and in $(\pi-x)^3$ at $x=\pi$ respectively. Such a function $f(x)$ must be the derivative with respect to x of a function $\psi_1(x^3)$ and also of $\psi_2[(\pi-x)^3]$. Neglecting a constant of integration, which may be included in either function, we must have $\psi_1(x^3)=\psi_2[(\pi-x)^3]$. Can we find a function single valued and analytic in x^3 at $x=0$ and also single valued and analytic in $(\pi-x)^3$ at $x=\pi$? If there is such a function, it will be invariant under the transformations $x'=\omega x$ and $\pi-x'=\omega(\pi-x)$, where $\omega=e^{2\pi i\,3}$; i. e., under the transformations

$$\begin{Bmatrix} x' = \omega x \\ x' = \omega x + (1-\omega)\pi \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} x' = \omega x \\ x' = x + (1-\omega)\pi \end{Bmatrix}.$$

Repetition of the first of these gives $x'=\omega^m x$, and of the second $x'=x+n(1-\omega)\pi$. Combination of these yields $x'=\omega^m x +n(1-\omega)\pi$. Repetition of this gives

$$x' = \omega^{m+p'}x + [q'\omega^m - q'\omega^{m+1} + n - n\omega]\pi,$$

or simply

$$x' = \omega^n x + (p + q\omega)\pi.$$

(Of course, m, n, p', q', p and q are integers.)

Consequently, x being an arbitrary point, the function must have the same value at the points

$$x + (p + q\omega)\pi, \quad \omega x + (p + q\omega)\pi, \quad \omega^2 x + (p + q\omega)\pi$$

that it has at x. A function having the same value at the points $x+(p+q\omega)\pi$ as at x is a doubly periodic function of primitive periods π and $\omega\pi$; call $g(x)$ such a function. Then $\psi(x)=g(x)+g(\omega x)+g(\omega^2 x)$ is a doubly periodic function of periods π and $\omega\pi$, and it is invariant under all of the above transformations. Hence $\psi(x)$ is a single valued function of x^3 and also of $(\pi-x)^3$, and $\psi'(x)$ can be expanded in both series.

The question of whether or not these series have a common region of convergence depends for its answer on the location of the poles of $\psi(x)$. It is clear that the poles may be situated so that the two triangles of convergence* have no common region in their interiors. On the other hand, it is not hard to put down the poles of $g(x)$ so that the triangles will have a region interior to both, as the following example shows.

Let $g(x)$ have a double pole (of course with zero residue) at the point $x=\pi/2+\pi i/2(3)^{1/2}$, and no other singularity in the parallelogram three of whose vertices are at 0, π, $\omega\pi$. Then $g(\omega x)$ and $g(\omega^2 x)$ will also have double poles at this point and nowhere else in this parallelogram. Hence $\psi(x)$ will have double poles at the points $x=\pi/2+\pi i/2(3)^{1/2}+p\pi+q\omega\pi$ and nowhere else, and the same is true of $\psi'(x)$. No singularity of $\psi'(x)$ is nearer the origin than the pole at $x=\pi/2+\pi i/2(3)^{1/2}$, and hence the circle of Theorem I may be given a radius slightly smaller than $\pi/3^{1/2}$ but greater than $\pi/2$. The circle of Theorem II may be given the same radius. Then these circles will have a region interior to both of them, which will contain a portion of the axis of reals, along which the series mentioned in both theorems will converge uniformly to $\psi'(x)$.

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* Hopkins showed in his special case that the region of uniform convergence is the interior of an equilateral triangle centered at $x=0$ and having one vertex on the positive axis of reals.