

AN ASSEMBLAGE-THEORETIC PROOF OF THE
EXISTENCE OF TRANSCENDENTALLY
TRANSCENDENTAL FUNCTIONS*

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1. *Introduction.* The first example of a transcendently transcendental function,† that is, of an analytic function not a solution of any algebraic differential equation, was given by Hölder,‡ who, in 1887, showed that the gamma function does not satisfy any such equation. Other investigations have followed Hölder's. In some of these, functional equations of different types are studied, to find which of their solutions satisfy algebraic differential equations. In others, e.g., in that of Hurwitz, on power series with rational coefficients§ and in that of Ostrowsky on Dirichlet series,|| certain analytic expressions are examined and conditions found under which they represent transcendently transcendental functions.

In the present note, the existence of transcendently transcendental functions is shown on an a priori basis, by a proof resembling that proof of the existence of transcendental numbers which is based on the countability of the algebraic numbers. The totality of algebraic differential equations has the same power as the totality of analytic functions, but there exists a countable set of the equations, namely, those with integral coefficients, whose solutions form the totality of the solutions of all algebraic differential equations. It follows that the solutions of the countable set of equations do not exhaust the analytic functions.

* Presented to the Society, December 29, 1926.

† This term is due to E. H. Moore, *Mathematische Annalen*, vol. 48 (1896), p. 49.

‡ *Mathematische Annalen*, vol. 28 (1887), p. 1.

§ *Annales de l'Ecole Normale Supérieure*, vol. 6 (1889), p. 327.

|| *Mathematische Zeitschrift*, vol. 8 (1920), p. 241.

Our result throws some light on the distribution of transcendently transcendental functions among the analytic functions. We prove that *it is possible to choose the coefficients in the series*

$$a_0 + a_1x + \dots + a_nx^n + \dots$$

so that the series does not satisfy formally any algebraic differential equation, with the following degree of freedom:

a_0 may be given any value not on a certain countable set;
 a_1 may then be given any value not on a certain countable set which depends on the choice of a_0 ;

a_n may be given any value not on a certain countable set which depends on the choices of a_0, \dots, a_{n-1} ;

Any set of a 's which do not increase too rapidly give a transcendently transcendental function.

2. *Algebraic Differential Equations.* Let $y(x)$ be any function satisfying an algebraic differential equation. Representing by y_p the p th derivative of y , we write the equation for y in the form

$$(1) \quad \sum c_{(i)} x^i y^{i_0} y_1^{i_1} \dots y_n^{i_n} = 0.$$

Here each c is a constant not zero. It is understood that the expressions $x^i \dots y_n^{i_n}$ are distinct from one another.

Now (1) states that, for the given function $y(x)$, the expressions $x^i \dots y_n^{i_n}$ are linearly dependent. Thus, if we set the wronskian of these expressions equal to zero, we shall have an algebraic differential equation for $y(x)$, usually of order greater than n , with integral coefficients. All that it is necessary to show is that the wronskian does not vanish identically in $x \dots y_n$. If it did, every function with n derivatives would satisfy an equation like (1), that is, an equation with the same expressions $x^i \dots y_n^{i_n}$ as appear in (1), with constants c not all zero. But because we can construct a function with any given values for itself and for its first n derivatives at any number of points, it

would follow easily that an equation like (1) exists, with coefficients c not all zero, which is an identity for x, \dots, y_n arbitrary. This is absurd. We know thus that $y(x)$ satisfies an algebraic differential equation with integral coefficients.*

3. *Transcendentally Transcendental Functions.* The algebraic differential equations with integral coefficients are countable. Let them be arranged in a sequence

$$(2) \quad e_1 = 0, \quad e_2 = 0, \quad \dots, \quad e_n = 0, \quad \dots$$

If the polynomial e_i vanishes when $x=h$, for all values of the variables other than x which appear in e_i , then e_i is divisible by $x-h$. As each e_i has only a finite number of factors of the form $x-h$, the polynomials of (2) have, collectively, only a countable set of such factors.

If we replace x in (2) by any value distinct from the h 's of the countable set just mentioned, the equations of (2) become non-identical equations which do not involve x ,

$$(3) \quad e'_1 = 0, \quad e'_2 = 0, \quad \dots, \quad e'_n = 0, \quad \dots$$

Of course, an e'_i may be a constant. This simply means that $e_i=0$ has no solution analytic for the chosen value of h .

We can now choose y anywhere, except on a countable set, in such a way as to reduce the equations of (3) to non-identical equations involving neither x nor y , and can continue in the same fashion for y_1 , etc. We find thus a set of values of our variables which satisfy no equation (2).

If the values chosen for x, y, y_1 , etc., are respectively h, k_0, k_1 , etc., the power series $\sum k_n(x-h)^n/n!$ will not satisfy formally any algebraic differential equation. The series $\sum k_n x^n/n!$ cannot satisfy such an equation either. This proves the result stated in the introduction.

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* Hurwitz, loc. cit., states without proof that a power series with rational coefficients which satisfies an algebraic differential equation, satisfies such an equation with integral coefficients. Very probably he had the more general result just proved in mind. In any case, the countability of the equations with integral coefficients plays no role in Hurwitz's work.