SCHLESINGER ON LEBESGUE INTEGRALS

Lebesguesche Integrale und Fouriersche Reihen. By L. Schlesinger and A. Plessner. Berlin and Leipzig, Walter de Gruyter Company, 1926. viii+229 pp.

This book is an outgrowth of a course of lectures by Professor L. Schlesinger at the University of Giessen in the winter semester of 1921–22 and was written with the cooperation of Dr. A. Plessner, who was at that time a student in the course, and later Professor Schlesinger's assistant. As the title indicates the subject is Lebesgue integrals and Fourier expansions from the point of view of Lebesgue's definition of integration.

Necessarily the first part is a study of the properties of real functions defined on point sets in a space of m dimensions. The first five chapters or 180 pages form a masterly exposition of the essential facts of real variable theory, particularly integration, and the sixth chapter of forty-two pages deals with Fourier series.

The headings give a sufficiently definite idea of the contents of each chapter. I. Fundamental concepts of points sets; II. Measure of point sets; III. Concerning functions of real variables; IV. The Lebesgue integral; V. Functions of one and two variables; VI. Fourier series.

Particular attention is directed to Chapter II in which the Borel-Lebesgue definition of measure is developed and compared with the older Peano-Jordan definition of content, and to Chapter IV where the Lebesgue integral is defined following Lebesgue's original geometric point of view. In the last section of this chapter (p. 129 ff.) attention is called to other useful definitions of the L-integral, due to Lebesgue himself, W. H. Young, J. Pierpont, and F. Riesz. This last definition will be discussed by the reviewer a little further on in greater detail, but at this point, it is well to note that mention should have been made along side of these of a two-way infinite series definition of a Lebesgue integral due to M. B. Porter.* By means of this definition the properties of the L-integral and the theorems of Lusin, Egoroff, and other important results are almost immediate.

Chapter VI contains an elegant and concise treatment of the most important newer results for Fourier series, notably those beginning with Lebesgue's *Leçons sur les Séries Trigonométriques*, 1906, which include the classic results of Riemann, Dini, Lipschitz, Dirichlet, Jordan, and others. The chapter closes with theorems on summability and convergence in the mean of Fejér, Cesàro, E. Fischer, and F. Riesz.

At the close of the preface is found a list of references to monographs, treatises, and encyclopedia articles whose content is closely related to the present text. Scattered throughout the book are to be found in footnotes, full references to the original memoirs bearing on all phases of the field.

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* This Bulletin, vol. 28 (1922), pp. 105–8.

The feature which, aside from the excellent form of the entire book, seems of foremost importance to the writer, receives no notice whatever in the table of contents and but one citation in the index. The number of references, however, is, in reality, six. These occur on pages 114, 123, 131, 194, 215, 221. This feature is the use of a sequence of step-functions (*Treppenfunktionen*) or *horizontal functions* in certain questions of convergence. The reviewer* has given the following definition:

A *horizontal function*, $h_n(x)$, *of index* n on an interval $I \equiv a \leqq x \leqq b$, associated with a subdivision of I into n subintervals, is one such that in the interior of the ith subinterval $h_n(x)$ is constant, i.e., $h_n(x) \equiv h_{in}$. The function need not be defined at the end points of the subintervals. Clearly $h_n(x)$ is represented for each value of n by n horizontal segments. Such functions are implicit in Riemann's and Lebesgue's (analytic) definition of an integral. In their explicit use, F. Riesz who called them "*fonctions simples*" seems to have been the first (references in the footnote, p. 131, in the present book under review). His use of them to give an alternate definition of an L-integral will be stated.

Concerning these horizontal functions, the following theorems are immediate. They will be stated without proof.

THEOREM I. *For every continuous function* $f(x)$ *on an interval* I, *there exists a sequence of horizontal functions* $f_n(x)$ *which has* $f(x)$ *for its limit function for every* x *in* I. *Furthermore, the sequence can be chosen so that the approach to the limit function will be monotone and uniform.*

THEOREM II. *For every bounded Riemann integrable function* $f(x)$ *on* I *there exists a sequence of horizontal functions* $f_n(x)$ *which has* $f(x)$ *for its limit function for every* x *in* I *save for a null set. Furthermore, the sequence can be chosen so that the approach to the limit function will be monotone.*

In the above two theorems, the set of horizontal functions may be built up in terms of a value of $f(x)$ in each subinterval. In particular, the maximum value (upper limit) or the minimum value (lower limit) may be used in each subinterval. We shall prove the following theorem used by Schlesinger for the simplest case.

THEOREM III. *For every bounded measurable function* $f(x)$ *on* I, *there exists a sequence of uniformly bounded horizontal functions* $f_n(x)$ *which has* $f(x)$ *for its limit function for every value of* x *on* I *save a null set.*

Proof. By hypothesis $f(x)$ has an indefinite L-integral

$$F(x) = \int_a^x f(t)\, dt,$$

where $F(x)$ has a bounded incremental ratio on I and $F'(x) = f(x)$ almost everywhere on I. Let $x_{in} = i(b-a)/n$ and let N be the number of sub-

* H. J. Ettlinger, *Note on a fundamental lemma concerning the limit of a sum*, this Bulletin, vol. 32 (1926), p. 69. See also H. J. Ettlinger, *On multiple iterated integrals*, American Journal, vol. 48 (1926), pp. 215-222.

intervals defined for every value of n by each pair of consecutive points $x_{ij}, i=1, 2, \cdots, j; j=1, 2, \cdots, n$. Let

$h_N(x) = [F(x_{i+1,N}) - F(x_{iN})]/[x_{i+1,N} - x_{iN}]$. Then $\lim_{n\to\infty} h_N(x) = f(x)$

for every x on I save a null set. Schlesinger proves this theorem for a bounded measurable function of m variables defined on a measurable point set.

The following theorem is in effect the equivalent of Riesz's definition of an L-integral. It may be regarded as a converse of Theorem III.

THEOREM IV. *If a uniformly bounded sequence of horizontal functions* $f_n(x)$ *on* I *has a limit function* $f(x)$ *almost everywhere on* I, *then* $f(x)$ *is bounded and measurable on* I, *i.e., is* L*-integrable on* I.

In terms of horizontal functions it is easy to establish the substantial equivalence of a *principle* used by the reviewer* on various occasions and named the Duhamel-Moore theorem,† and a theorem due to Lebesgue‡ on the integral of the limit of a sequence of functions. The Duhamel-Moore theorem (DM) may be stated for an interval I as follows:

HYPOTHESIS. 1. *The set of* n *non-overlapping subintervals* I_{in}, $i=1, 2, \cdots, n$, *of length* l_{in} *are obtained by subdividing the interval* I. *The sets of numbers* r_{in}, r'_{in} *are such that* $|r_{in} - r'_{in}|$ *is uniformly bounded.*

2. *If* P *is a point of* I *then* $\lim_{n\to\infty}(r_{iP_n n} - r'_{iP_n n}) = 0$ *almost everywhere on* I.

3. $\operatorname{Lim}_{n\to\infty} \sum_1^n r_{in} l_{in}$ *exists.*

CONCLUSION

1. $\operatorname{Lim}_{n\to\infty} \sum_1^n r'_{in} l_{in}$ *exists.*

2. $\operatorname{Lim}_{n\to\infty} \sum_1^n r_{in} l_{in} = \operatorname{Lim}_{n\to\infty} \sum_1^n r'_{in} l_{in}$.

Lebesgue's theorem (L) may be stated as follows:

HYPOTHESIS. 1. $f_n(x)$ *is a uniformly bounded sequence of integrable functions on* I.

2. $\operatorname{Lim}_{n\to\infty} f_n(x) = f(x)$ *almost everywhere on* I.

CONCLUSION

1. $f(x)$ *is bounded and integrable on* I.

2. $\operatorname{Lim}_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$.

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* See my paper entitled *On multiple iterated integrals*, loc. cit., for complete references.

† R. L. Moore, *On Duhamel's theorem*, Annals of Mathematics, (2), vol. 13 (1912), pp. 162–3.

‡ *Leçons sur l'Intégration*, etc. (Borel monograph.) Paris, Gauthier-Villars, 1904, p. 114.

By the use of Theorems III and IV, we derive (DM) from (L). Let $r_{in} - r'_{in}$ define a sequence of horizontal functions on I. The limit function is obviously a null function whose integral is zero. Conclusions 1 and 2 of (DM) follow immediately from (L).

Conversely we derive (L) from (DM). By Theorem IV, $f(x)$ is bounded and integrable in I. Let $h_n(x)$ be a set of horizontal functions on I with respect to I_{in} approaching $f(x)$ as a limit function almost everywhere on I. Identify $r_{in} = h_{in}$. Also let $H_n^{(k)}(x)$ be a horizontal function of index n on I associated with I_{in} and having $f_k(x)$ as a limit function almost everywhere on I. Identify $r'_{in} = H_{in}^{(n\,k)}$ and Conclusion 2 of (L) follows from (DM).

It is worthwhile emphasizing the role which horizontal functions, together with the principle isolated by R. L. Moore, are destined to play in the theory of functions of a real variable. They may be made the basis for a concise and elegant treatment of the greater part of the theory. The bulk of a book like Hobson's *Theory of Functions of a Real Variable*, volume I, may be reduced to one-third or less by their use. Schlesinger in the book under review has made a beginning in this direction.

H. J. ETTLINGER

FUBINI AND ČECH, PROJECTIVE DIFFERENTIAL GEOMETRY

Geometria Proiettiva Differenziale, Vol. I. By G. Fubini and E. Čech. Bologna, N. Zanichelli, 1926. 388 pp.

"Un nuovo indizio dei sentimenti fraterni che vanno sempre più legando fra loro i vari rami della matematica!"

These words of Segre were chosen by Wilczynski to be inscribed on the title page of his prize memoir *Sur la théorie générale des congruences*. Thus the founder of the American school of projective differential geometry indicated that he was studying the projective differential properties of a geometric configuration by the means of the invariants and covariants of a completely integrable system of linear homogeneous partial differential equations under a certain continuous group of transformations, in the sense of Lie. The same sentiment would be no less appropriate as a motto for the new book by Fubini and Čech, since these distinguished protagonists of the Italian school of projective differential geometry define a configuration by means of differential forms, after the manner of Gauss, and employ the absolute calculus of Ricci.

Those who know the absolute calculus only as it is used in the theory of relativity will be interested to see this geometric application of it. And those who know only Wilczynski's method of attacking a problem in projective differential geometry will be eager to learn this new theory. Wilczynski's method is particularly adapted to certain types of problems and has a power and elegance of its own. But it has some inconveniences. For instance, certain calculations become quite laborious, which are accomplished more easily and efficiently by the tensor analysis.

The first volume of the treatise before us is dedicated to the dean of differential geometers, Luigi Bianchi, and is devoted to the geometry of