A CONNECTED AND CONNECTED IM KLEINEN POINT SET WHICH CONTAINS NO PERFECT SUBSET

BY B. KNASTER AND C. KURATOWSKI

1. *Introduction.* Professor R. L. Moore has shown in this Bulletin (vol. 32, p. 331) that there exist point sets connected and connected im kleinen* which contain no arc. We shall prove in this paper the existence of such a set containing no *perfect* subset. The set has the additional property that it is contained in a *regular curve* (in the sense of K. Menger)†.

2. *The Sierpinski Regular Curve.* Let R be the Sierpinski regular curve,‡ defined as follows. Let T be an equilateral triangle. Divide T in 4 equal triangles. Let T_0, T_1, T_2 denote those three triangles which have a common vertex with T. Similarly divide each of the triangles T_0, T_1, T_2 in 4 equal triangles and let T_{00}, T_{01}, T_{02}, T_{10}, $\cdots$, T_{22} denote those having a common vertex with T_0 or T_1 or T_2; and so on *ad inf.*

The point set formed by the boundaries of all the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}(\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_k = 0, 1, \text{ or } 2)$ and all their limit points is the regular curve R.

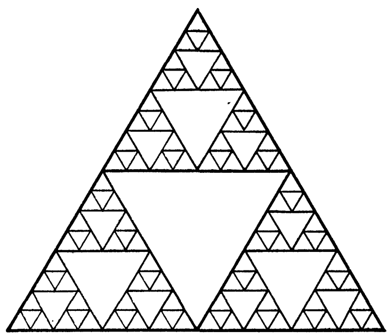
---

* A point set M is said to be *connected im kleinen* (or to be regular) if, for every point p and every positive number e, there exists a positive number d such that if x is any point of M at a distance from p less than d then x and p both lie in some connected subset of M of diameter less than e.

† A continuum C is called a regular curve if, for every positive number e, C can be expressed as the sum of a finite number of continua each of diameter less than e, each pair of continua having at most a finite number of points in common (see K. Menger, Mathematische Annalen, vol. 95 (1925), p. 300). Every regular curve is a continuous curve whose every subcontinuum is a continuous curve (see H. M. Gehman, Annals of Mathematics, vol. 27 (1925), p. 42).

‡ Prace Matematyczno-Fizyczne, vol. 27 (1915). The Sierpinski curve R contains three points of degree 2, a countable set of points of degree 4, the remainder being composed of points of degree 3. See also Comptes Rendus, vol. 160 (1915), p. 302.

3. *Properties of the Sierpinski Curve.* The complementary set of R is composed of a countable set of regions. Let I_0 denote the unbounded region (exterior to T) and $I_1, I_2, \cdots, I_n, \cdots$ the bounded regions (for $n>0$, I_n forms the interior of a triangle). Let B_n denote the boundary of I_n and V the countable point set of all the vertices of the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}$. The following properties of the curve R are to be noticed:



PROPERTY 1. *Given a positive integer k, no point of R belongs to more than two of the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}$.*

PROPERTY 2. *If $m \neq n$, then $\bar{I}_m \cdot \bar{I}_n \subset V$**.

PROPERTY 3. *Given two points a and b of R and an index n, there exists a subcontinuum C of R containing a and b and such that $C \cdot B_n \subset V+a+b$.*

In a similar manner we have

PROPERTY 4. *Given a triangle $T_{\alpha_1\alpha_2\cdots\alpha_k}$ and an index n, any two points a and b of $R \cdot T_{\alpha_1\alpha_2\cdots\alpha_k}$ may be joined by a subcontinuum C of $R \cdot T_{\alpha_1\alpha_2\cdots\alpha_k}$ such that $C \cdot B_n \subset V+a+b$.*

4. *Two Lemmas.* We shall now prove two lemmas.

LEMMA I. *If K is a bounded continuum such that $K \cdot V=0$ and there exists no index n such that $K \subset \bar{I}_n$, then the set $K \cdot R$ contains a perfect subset.*

* The symbol $\bar{X}$ denotes the set $X+$ all its limit points. The symbol $X \subset Y$ means that X is contained in Y.

PROOF. It is evident that

$$K = K \cdot R + K \cdot I_0 + K \cdot I_1 + \cdots + K \cdot I_n + \cdots ;$$

hence

$$(1) \qquad K = K \cdot R + K \cdot \bar{I}_0 + K \cdot \bar{I}_1 + \cdots + K \cdot \bar{I}_n + \cdots.$$

Suppose $K \cdot R$ does not contain any perfect subset. As $K \cdot R$ is closed, it follows that $K \cdot R$ is a finite or countable point set.

Let S_n denote the set $K \cdot \bar{I}_n$ and let Q denote the set $K \cdot R-(S_0+S_1+ \cdots +S_n+ \cdots)$. Hence Q is a finite or countable set of points: $p_1, p_2, \cdots$. It follows by (1) that

$$(2) \quad K = p_1 + p_2 + \cdots + S_0 + S_1 + \cdots + S_n + \cdots .$$

The right-hand side of the identity (2) is composed of mutually exclusive sets, since by Property 2, if $m \neq n$, then $K \cdot \bar{I}_m \cdot K \cdot \bar{I}_n \subset K \cdot V = 0$. But this contradicts a theorem of Sierpinski's* to the effect that if K is a sum of a finite ($\geqq 2$) or countable number of mutually exclusive closed point sets, then K is not a bounded continuum. Thus the existence of a perfect subset of $K \cdot R$ is established.

LEMMA II. *If Z is a subset of R such that each perfect subset of R contains a point belonging to Z, then $Z+V$ is connected and connected im kleinen.*

PROOF. Suppose that the set $Z+V$ is not connected. Then† there exist two points a and b of $Z+V$ and a bounded continuum K which separates a from b and is such that $K \cdot (Z+V)=0$. Therefore

$$(3) \qquad K \cdot (V + a + b) = 0.$$

Since $K \cdot Z=0$, it follows, by hypothesis, that the set $K \cdot R$ does not contain any perfect subset. By Lemma I, there exists an index n such that $K \subset \bar{I}_n$. As $\bar{I}_n=I_n+B_n$ and $R \cdot I_n=0$, it follows that

* Tôhoku Mathematical Journal, vol. 13 (1918), p. 300.

† See our paper *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 233, Theorem 37.

$$(4) \qquad K \cdot R \subset B_n.$$

By Property 3, the points a and b may be joined by a subcontinuum C of R such that

$$(5) \qquad C \cdot B_n \subset V + a + b.$$

Since $K \cdot C = K \cdot R \cdot C$, it follows from (4) and (5) that

$$K \cdot C \subset B_n \cdot C \subset V + a + b.$$

Hence by (3) $K \cdot C = 0$, contrary to the assumption that K separates a from b. Thus the supposition that $Z + V$ is not a connected point set leads to a contradiction.

Now let H denote any one of the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}$. Each perfect subset of $R \cdot H$ has a common point with $Z \cdot H$. By an argument similar to that used above it may be proved (with the help of Property 4 instead of Property 3) that the set $(Z+V) \cdot H$ is connected. It follows by Property 1 that $Z+V$ is connected im kleinen.

5. *Conclusion*. We may now state the following theorem.

THEOREM. *There exists in the regular curve R of Sierpinski's a connected and connected im kleinen point set which contains no perfect subset.*

PROOF. By a theorem due to F. Bernstein* the plane may be decomposed into two mutually exclusive subsets E and F such that each perfect set contains points of both of them. It follows that each perfect subset of R has a common point with $E \cdot R$. By Lemma II the set $M = E \cdot R + V$ is connected and connected im kleinen.

The set M contains no perfect subset. For suppose P is a perfect subset of M. As the set V is countable, $P - V$ contains a perfect subset P_1. Hence $P_1 \subset E$, contrary to the assumption that P_1 has a common point with F. Thus M is a connected and connected im kleinen point set which contains no perfect subset.

THE UNIVERSITY OF WARSAW

---

* Leipziger Berichte, vol. 60 (1908).