SINGULARITIES OF THE HESSIAN*

BY T. R. HOLLCROFT

1. *Introduction.* It has been proved† that when a curve f has no point singularities its Hessian H has no point singularities. Then point singularities occur on H when and only when f has point singularities. Moreover, singular points of H can occur only at those points which are singular points of f.

The number of intersections of f and H at any singularity of f is

$$6\delta_1 + 8\kappa_1 + \iota_1$$

where δ_1, κ_1, ι_1 are the numbers of nodes, cusps, and inflections respectively contained in the singularity of f. A given singularity of f needs but to be resolved and the number of intersections of f and H are thus found without reference to H. In order for this number of intersections to occur, there must be a singularity of H at this point, but except for cusps and simple multiple points with distinct tangents these singularities of H have not been investigated.

The purpose of this paper is to explain geometrically how the intersections of f and H at a given singularity of f occur. The principal problem involved is to find the singularity of H corresponding to a given singularity of f.

2. *Simple Multiple Points.* It has long been known that at a simple r-fold point of f with r distinct tangents, H has a $(3r-4)$-fold point, r of whose tangents coincide, one each, with r tangents of the r-fold point on f; also that at a cusp of f, H has a triple point two of whose tangents coincide with the cuspidal tangent.

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† A. B. Basset, *On the Hessian, the Steinerian, and the Cayleyan*, Quarterly Journal, vol. 47 (1916), p. 227.

Any number of cusps up to and including $r-1$ may occur in a simple r-fold point. The occurrence of κ_1 cusps in such a multiple point causes κ_1+1 tangents at that point to become consecutive. Also each cusp implies two additional intersections of f and H at this point.

Consider that f has an r-fold point at the origin O, the tangents at which are given by the equation $u_r=0$. Of the $3r-4$ tangents to H at O, r have the equation $u_r=0$ and the remaining $2(r-2)$ have the equation

$$T \equiv \frac{\partial^2 u_r}{\partial x^2} \cdot \frac{\partial^2 u_r}{\partial y^2} - \left(\frac{\partial^2 u_r}{\partial x\, \partial y}\right)^2 = 0.*$$

If the singularity of f at O contains $c-1$ cusps, it has c consecutive tangents. The tangents to f at O are now given by

$$u_r \equiv (ax + by)^c u_{r-c} = 0.$$

For this form of u_r, the equation T becomes

$$T \equiv (ax + by)^{2(c-1)} U_{2(r-c-1)} = 0,$$

where $U_{2(r-c-1)}$ is a function of $ax+by$, u_{r-c} and their first and second partial derivatives. Then each cusp in the r-fold point O of f causes two of the $2(r-2)$ additional tangents to H at O to become consecutive with the two consecutive tangents common to H and f.

If, in the above, $c=r$, the expression for T is identically satisfied so that it no longer is the equation of the additional tangents to H at O. This shows that the above conclusions are justified when and only when the number of cusps in the r-fold point of f does not exceed $r-2$. When it contains $r-2$ cusps, there are but two distinct tangents to f at P and these two are the only tangents to H at P. The $(3r-4)$-fold point on H now contains $3(r-2)$ cusps. If a general $(3r-4)$-fold point, it could contain one more cusp, causing all its tangents to become consecutive, but this special Hessian multiple point can not have this additional cusp, that is, a simple

* Hilton, *Plane Algebraic Curves*, 1920, Ex. 11, p. 103.

multiple point on a Hessian can not consist of a single superlinear branch.

The r-fold point P on f, however, may contain $r-1$ cusps and consist of a single superlinear branch. At P, H now has a $3(r-1)$-fold point with $2(r-1)$ of its tangents consecutive with the single tangent to f at P and the remaining $r-1$ tangents distinct from this and from each other. The well known triple point of H at a cusp of f is a special case of this for $r=2$.

Consider the r-fold point of f first without cusps and let nodes be changed into cusps one by one up to and including the maximum. Each cusp up to and including the $(r-2)$th causes three nodes of the $(3r-4)$-fold point of H to be converted into three cusps. The genus of H is not affected by this exchange so long as the number of cusps of f at P does not exceed $r-2$. When the $(r-1)$th node is changed into a cusp, nothing different from usual happens on f, but something very different occurs on H. The multiplicity of P on H is increased by unity causing the addition of $3r-4$ nodes and a corresponding decrease of $3r-4$ in the genus of H and, in addition, $r-2$ cusps in the multiple point of H at P are changed back into nodes causing $r-2$ branches of H at P that had been coincident to become distinct.

If $r=n-1$, H is of order $3(n-2)$ with a $(3n-7)$-fold point at P. In this case and in this case only can the Hessian be rational. If $r=n-1$, H is a proper curve when and only when the number of cusps of f at P does not exceed $n-3$. If the $(n-1)$-fold point of f contains $n-2$ cusps, H degenerates into $3(n-2)$ lines through P of which $2(n-2)$ coincide with the single tangent to f at P. A well known special case of this is the Hessian of a cuspidal cubic.

3. *Compound Singularities.* When the singularity of f at P is compound, the singularity of H at P is also compound.

Any number s of consecutive r-fold points at P on f with r distinct branches determines at P, on H, s consecutive $3(r-1)$-fold points with $3(r-1)$ distinct branches and with the same tangent as that to f at P. There are, then, at P, r branches of

f, each of which has contact of order s with each of the $3(r-1)$ branches of H that pass through P.

Any series of consecutive multiple points of orders $r_1, r_2, r_3, \cdots$ where $r_1 > r_2 \geqq r_3 \geqq \cdots$ and $r_1 > r_2+1$ gives rise to a series of consecutive multiple points on H of the respective orders $3r_1-4$, $3(r_2-1)$, $3(r_3-1)$, $\cdots$ and through the same respective penultimate points and therefore with the same tangent. If, however, $r_2 \leqq r_1 \leqq r_2+1$, the series on H are of the respective orders $3(r_1-1)$, $3(r_2-1)$, $3(r_3-1)$, $\cdots$.

In case $r_1 = r_2$, the $3(r_1-1)$ branches of H all have the same tangent at P as the r_1 branches of f. In case $r_1 = r_2+1$, at P on f there are r_2 branches with the same tangent and one branch with a distinct tangent. At P on H there are $3r_1-4$ branches that have the tangent common to the r_2 branches of f and the remaining branch of H has the same tangent as the remaining branch of f.

Since any series of consecutive multiple points of f gives rise to a series of consecutive multiple points of H through the same penultimate points respectively, the order of nearness* of any two consecutive points of H is always equal to the order of nearness of the two corresponding consecutive multiple points of f.

Any series of consecutive multiple points of orders $r_1, r_2, r_3, \cdots$ where $r_1 \geqq r_2 \geqq r_3 \geqq \cdots$ may contain any number of cusps up to and including r_1-1. For each node changed into a cusp in the part of the r_1-fold point involved in the consecution or in any of the other consecutive points, a bitangent of the singularity is changed into a stationary tangent. Then if the number of cusps k is such that

$$k \leqq r_2 - 1$$

the singularity contains k inflections, but if

$$r_1 - 1 \geqq k \geqq r_2 - 1$$

* See F. Enriques, *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, vol. 2, pp. 404–408.

the singularity contains r_2-1 inflections provided that the cusps are considered as being introduced so far as possible into that part of the singularity involved in the consecution. Then the introduction of $k \leqq r_2-1$ cusps causes the addition of $3k$ intersections of f and H at this point, $2k$ for the cusps that replace k nodes and k for the k inflections that replace k bitangents. The introduction of k cusps into the portion of the r_1-fold point not involved in the consecution causes the addition of only $2k$ intersections as for simple multiple points.

If the consecutive multiple points of f at P contain any number k of cusps all of which are involved in the consecution, the consecutive multiple points of H at P contain $3k-1$ cusps. All the branches of both f and H at P (except the simple branches of the r_1-fold point of f and the $(3r_1-4)$-fold point of H in case $r_1>r_2$) have the same tangent whether the points contain cusps or not. The change of k nodes into k cusps on f causes $k+1$ of the branches to coincide, while on H the $3k-1$ cusps cause the coincidence of $3k$ branches. H and f must now have $3k$ additional intersections at P. None of these are accounted for by the mere coincidence of the branches, but all are accounted for by the fact that the superlinear branches of f and H have closer contact than had each of the component branches when distinct. If there are s consecutive r-fold points of f at P, each branch of f has s-point contact with each branch of H at P. When the multiple points of f contain k cusps, however, each of the $k+1$ partial branches of the superlinear branch of f at P has $[(s+1)/(k+1)]$-point contact with each of the $3k$ partial branches of the superlinear branch of H at P.

A flecnode on f gives rise to a flecnode on H with the same stationary and simple tangents respectively. Also a biflecnode on f gives rise to a biflecnode on H with the same two stationary tangents. In general, if any of the tangents at an r-fold point of f are stationary tangents, these are also stationary tangents to H at this point.

4. *Line Singularities.* Simple line singularities of f cause on H either no singularities or simple line singularities. If the

line singularity of f be composed of only bitangents, that is, if all its contacts are distinct, H avoids both this line and its contacts and has no singularity caused by such a multiple line of f. Moreover, a multiple line of class ρ of f may contain any number up to $(\rho+1)/2$ inflections, no more than one at each contact except that one point of contact may contain two inflections, and H will have no line singularity. It is only when more than two inflections occur at one contact or more than one inflection at two or more contacts of a multiple line p of f that H has p as a multiple line.

In general, if f has a multiple line p of class ρ and q is the number of contacts of p with f that contain two or more inflections and i_j is the number of inflections contained in any one of these q contacts, then p is a multiple tangent of H of class

$$\rho_1 \equiv \sum_{j=1}^{q} i_j - q.$$

Since the number of inflections of a multiple line of class ρ with q distinct contacts can not exceed $\rho-q$, there is an upper limit to the class ρ_1 of the line p of H

$$\rho_1 \leqq \rho - 2q.$$

The equality sign holds only when no contacts of p with f contain less than two inflections. It is also apparent that the fewer the number of distinct contacts, the larger the class of the multiple line of H may be. Thus the maximum class of a multiple line of H that results from a multiple line of class ρ of f is $\rho-2$. This occurs when and only when p contains $\rho-1$ stationary tangents to f, in which case $q=1$. In this case, the multiple line p contains $\rho-3$ stationary tangents to H and all its contacts are at one point.

5. *Singularities that a Hessian Cannot Have.* In the preceding sections, the multiple points and lines that a Hessian can have were discussed. There are, however, certain singular points that a Hessian can not have under any circumstances. These include all simple or compound double points except

distinct nodes, multiple points of order $3a+1$ for a any integer and simple multiple points with all tangents consecutive.

There are no such restrictions on multiple lines of the Hessian.

6. *The Hessian as a Jacobian.* Since the Hessian of f is the Jacobian of the net of first polars of f, the singularity P of H caused by a given singularity P of f is the singularity of the Jacobian of the net of first polars of f corresponding to this basis point P on the first polars. Since an r-fold point P of f causes a $(3r-4)$-fold point P of H and an $(r-1)$-fold point P on each first polar, then H as the Jacobian has a $(3i-1)$-fold point at an i-fold basis point of the net. Since this result is general, it might be inferred that the singularities of the Jacobian corresponding to any kind of basis point could be now found by finding the kind of basis point on the first polars determined by a given singularity of f. However, the result is general only for simple multiple points with distinct branches. If the curves of the net have contact of any given order with each other at P, the nature of the singularity P on the Jacobian can not be predicted unless more is known about the net. For example, simple contact in a first polar net may result from a cusp on f or a tacnode on f. In these cases, the singularity on the Jacobian is respectively a triple point two of whose tangents coincide with the cuspidal tangent or two consecutive triple points whose tangent coincides with the tacnodal tangent. But a simple contact in a net of otherwise general curves causes on the Jacobian a triple point with three distinct tangents one of which coincides with the common tangent of the curves of the net.

For $n>3$, first polar nets are not the most general nets of order $n-1$ and for all values of n, if f has singularities, first polar nets are very highly specialized. For this reason (except in the case of a simple multiple point with distinct tangents) general results for the singularity of the Jacobian at a given singularity of the net can not be obtained by regarding the Hessian as the Jacobian of the net of first polars.

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