NUCLEAR POINTS IN THE THEORY OF ABSTRACT SETS*

BY W. SIERPINSKI

1. *Introduction.* E. W. Chittenden has published an interesting note† in which he gave necessary and sufficient conditions that a set E contained in a class (V) (of Fréchet) be perfectly compact, or perfectly self-compact.‡

Chittenden proves the following theorems:

THEOREM I. *If an infinite set E is perfectly compact, E determines at least one nuclear point.*

COROLLARY. *Every set E which is perfectly self-compact contains a nuclear point.*

THEOREM II. *If every infinite subset of a set E of points of the space P determines a nuclear point then E is perfectly compact.*

THEOREM III. *A necessary and sufficient condition that a set E be perfectly compact is that every infinite subset of E determine at least one nuclear point.*

COROLLARY. *A necessary and sufficient condition that every compact set E be perfectly compact is that every infinite compact set E possess at least one nuclear point.*

The proofs of Chittenden are based on the following assumption (§ 3): "Let Q be an aggregate of power μ and of elements q. Of the transfinite ordinals Ω of which the aggregate of all ordinals $\alpha<\Omega$ has the power μ there is a least, Ω_0. Let

* Presented to the Society, September 9, 1926.

† This BULLETIN, vol. 30 (1924), p. 511.

‡ For the definitions of the terms *class* (V), *monotone sequence, compact, perfectly compact, perfectly self-compact, limit point, nuclear point,* see E. W. Chittenden, loc. cit. §§2, 4. A topological space in which every infinite set determines a nuclear point is called by P. Alexandroff and P. Urysohn a *bicompact* space (see MATHEMATISCHE ANNALEN, vol. 92 (1924), p. 260).

(1) $q_1, q_2, q_3, \cdots, q_n, \cdots, q_\omega, \cdots, q_\alpha, \cdots, \qquad (\alpha < \Omega_0)$

represent a 1-1 correspondence between the aggregate Q and the aggregate of all ordinal numbers $\alpha < \Omega_0$. *If a sequence of ordinals $\beta < \Omega_0$ is determined so that for every α there is a $\beta > \alpha$, then the β's form a series of the ordinal type Ω_0 and the aggregate of all such ordinals β is of power μ.*"

This assumption is equivalent to the assumption that μ is a *regular* aleph. Since not every aleph is regular, the theorems of Chittenden are proved only for the sets E whose power is a regular aleph.

In this note I shall examine the question: Are the theorems of Chittenden true for *every* set E without any restriction?

2. *Theorem* I *may be False.* Theorem I may be false for a set E whose power is an irregular aleph, e. g., for

$$\aleph_\omega = \sum_{n=0}^{\infty} \aleph_n.$$

Let E be the set of all ordinals $\xi < \omega_\omega$ (ω_ω denoting the least ordinal of power $\aleph_\omega$). Let H be the set of all the sequences $\xi_1, \xi_2, \xi_3, \cdots$ made up of different elements of E. Let $P = E + H$; the neighborhoods are defined as follows.

Let ξ be any element of E and let $\lambda < \xi$; the set of all ordinals η such that $\lambda < \eta \leqq \xi$ is called a neighborhood of ξ. Let $\alpha = (\xi_1, \xi_2, \xi_3, \cdots)$ be any element of H; we call neighborhood of α any set composed of α and $\xi_n, \xi_{n+1}, \xi_{n+2}, \cdots$ $(n = 1, 2, 3, \cdots)$.

Evidently every neighborhood of an element belonging to E contains less than $\aleph_\omega$ elements of P, and every neighborhood of an element belonging to H contains $\aleph_0$ elements of P. Hence every neighborhood contains less than $\aleph_\omega$ elements. Thus, there does not exist any nuclear point of the set E (as the power of E is $\aleph_\omega$).

I shall prove that E is perfectly compact. Let $S = \{G\}$ be any sequence of decreasing subsets of E. Let S_0 be a well-ordered sequence "confinal with S" (i. e. S_0 is contained in S and for each set G belonging to S there exists a set of S_0 contained in G).* The sequence S_0 may be supposed of the least possible ordinal number. Moreover, we may suppose that S does not contain a last element. Hence the ordinal number of S_0 is an "initial" ordinal number ω_n, where n is either 0 or a natural number. Evidently we may assume $S=S_0$. Consider two cases.

CASE 1. $n=0$. Hence $S=(G_1, G_2, G_3, \cdots)$. For each n, we may select an element α_n of G_n that does not belong to G_{n+1}. The aggregate $\alpha=(\alpha_1, \alpha_2, \alpha_3, \cdots)$ is an element of H and is a limit point of any sequence $\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \cdots$. As each element of the sequence $\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \cdots$ belongs to G_n, α is a limit point of G_n $(n=1, 2, 3, \cdots)$. Hence α belongs to G' for each G of S.

CASE 2. $n\geqq 1$. In this case the set $S=\{G_\xi\}$ is not enumerable. Select, for each ξ, an element p_ξ of G_ξ that does not belong to $G_{\xi+1}$ (hence p_ξ does not belong to G_η if $\eta>\xi$). The set Q of all p_ξ is of power $\aleph_n$.

Let E_k be the set of the ordinals $\xi\leqq\omega_k$. Hence $E=E_1+E_2+\cdots$. Hence there exists an index k such that Q contains $\aleph_n$ elements of E_k (otherwise Q would contain $\leqq\aleph_{n-1}$ elements of each E_k and Q would be of power $\leqq\aleph_{n-1}\cdot\aleph_0=\aleph_{n-1}<\aleph_n$, contrary to hypothesis). Let these elements be

$$(2)\qquad p_{\xi_1}, p_{\xi_2}, \cdots, p_{\xi_\omega}, \cdots, p_{\xi_\alpha}, \cdots, \qquad (\alpha<\omega_n).$$

Let γ be the least number having the property that there exists in the sequence (2) a set of power $\aleph_n$ of numbers $\leqq\gamma$. As the sequence (2) is contained in E_k, $\gamma\leqq\omega_k$. Now, if $\lambda<\gamma$ there is set of power $\aleph_n$, composed of elements of the sequence (2) and contained between λ and γ. It follows that, if $\alpha<\omega_n$, each neighborhood of γ contains at least one element of G_{ξ_α}. As the sequence $G_{\xi_1}, G_{\xi_2}, \cdots$, $G_{\xi_\alpha}, \cdots$ $(\alpha<\omega_n)$ is confinal with $G_1, G_2, \cdots, G_\xi, \cdots$, it follows at once that each neighborhood of γ contains an element of G_ξ. Hence γ belongs to G'_ξ for each $\xi<\omega_n$.

* Hausdorff, *Grundzüge der Mengenlehre*, p. 132, Leipzig, Veit, 1914.

Thus E is perfectly compact and Theorem I is, in this case, false. It may be proved by a similar argument that for any irregular aleph there exists a set E for which the Theorem I does not hold true.

3. *The Corollary of Theorem* I *holds True for any Set E.* Since Chittenden gave a proof of the corollary in case the power of E, call it $\aleph_\alpha$, is a regular aleph, I shall suppose that $\aleph_\alpha$ is irregular. It follows that there exists an ordinal $\beta<\alpha$ (of the second kind) and a transfinite sequence of increasing ordinals $\alpha_\xi<\alpha$ $(\xi<\beta)$ such that $\lim_{\xi<\beta}\alpha_\xi=\alpha$ and

(3) $$\aleph_\alpha = \sum_{\xi<\beta} \aleph_{\alpha_\xi}.$$

Suppose E is a perfectly self-compact set that does not contain any nuclear point. Then there exists for each element p of E a neighborhood $V(p)$ such that the product $E \cdot V(p)$ is of power $< \aleph_\alpha$. Let

(4) $$p_1, p_2, p_3, \cdots, p_\omega, \cdots, p_\kappa, \cdots \qquad (\lambda < \omega_\alpha)$$

be the sequence (of the least possible type) constituted of all the elements of E.

Let $\xi<\beta$; let T_ξ be the sum of all the sets $E \cdot V(p_\lambda)$ such that $E \cdot V(p_\lambda)$ is of power $\leqq \aleph_{\alpha_\xi}$ and $\lambda<\omega_{\alpha_\xi}$. Thus T_ξ is obtained by adding together $\leqq \aleph_{\alpha_\xi}$ sets each of power $\leqq\aleph_{\alpha_\xi}$. It follows that the power of T_ξ is $\leqq \aleph_{\alpha_\xi}^2=\aleph_{\alpha_\xi}<\aleph_\alpha$ (since $\xi<\beta$ implies $\alpha_\xi<\alpha$). As E is of power $\aleph_\alpha$ there must exist in E a point q_ξ that does not belong to T_ξ. Now as the numbers α_ξ $(\xi<\beta)$ are increasing, it follows from the definition of T_ξ that

(5) $$T_\xi \subset T_\eta \quad \text{if} \quad \xi \leqq \eta < \beta.$$

Let S_ξ denote the set of points q_η such that $\xi\leqq\eta<\beta$. The sets S_ξ are decreasing. Now, E is perfectly self-compact. Hence the sets S_ξ or their derived sets S_ξ' have a common point q. As q belongs to E there is an ordinal λ such that $q=p_\lambda$. As $\lambda<\omega_\alpha$ and $\lim_{\xi<\beta}\alpha_\xi=\alpha$, $\lim_{\xi<\beta}\omega_{\alpha_\xi}=\omega_\alpha$, there exists an ordinal $\rho<\beta$ such that $\lambda<\omega_{\alpha_\rho}$. Now, since the set $E \cdot V(p_\lambda)$ is of power $< \aleph_\alpha$ there exists an ordinal $\zeta<\beta$ such that $E \cdot V(p_\lambda)$ is of power $\leqq \aleph_{\alpha_\zeta}$. Let $\xi=\rho$ in case $\rho \geqq \zeta$ and $\xi=\zeta$ in case $\rho<\zeta$. Thus: $\xi<\beta$, $\lambda<\omega_{\alpha_\xi}$ and $E \cdot V(p_\lambda)$ is of power $\leqq \aleph_{\alpha_\xi}$. Hence p_λ belongs to T_ξ. It follows from the definition of the sequence $\{q_\eta\}$ and from (5) that q_η does not belong to T_ξ if $\eta \geqq \xi$. Hence q_η does not belong to $V(q)$ if $\eta \geqq \xi$, since the set $E \cdot V(q)=E \cdot V(p_\lambda)$ is contained in T_ξ. Therefore $S_\xi \cdot V(q)=0$, contrary to the hypothesis that q is a common point either of all the sets S_ξ or of all the S_ξ'.

Consequently, E must contain a nuclear point.

4. *Theorem* II *holds True for any Set E.* This may be proved as follows. Let S be a monotonic sequence of subsets of E. It may be assumed (cf. § 2) that S is a well-ordered series of decreasing sets G_α and that the ordinal number of S is an initial number ω_γ. Let p_α be in G_α but not in $G_{\alpha+1}$. The set H of all p_α is of power $\aleph_\gamma$.

By hypothesis, H determines a nuclear point q. Hence each neighborhood V of q contains $\aleph_\gamma$ points of H. It follows that V contains at least one point of every G_α. Therefore q belongs to all the G_α'. Consequently E is perfectly compact.

5. *Conclusions.* Thus Theorem II holds true for any E, but Theorem I may be false if the power of E is an irregular aleph. If follows at once that although *the condition stated in Theorem* III *is sufficient, it is not necessary.*

Consequently *the condition stated in the Corollary of Theorem* III *is sufficient.* However, *it is not necessary.*

For, let K be the set of all ordinals $\xi<\omega_\omega$ and call neighborhood of ξ any set of all ordinals η such that $\lambda<\eta\leqq\xi$ (where $\lambda<\xi$). By an argument analogous to that of § 2, it is easy to see that every compact subset E of K is perfectly compact. Evidently there does not exist in K any nuclear point.

THE UNIVERSITY OF WARSAW