NOTE ON A THEOREM OF KEMPNER CONCERNING TRANSCENDENTAL NUMBERS*

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The theorem in question is the following one:†

Let a be an integer greater than 1, p/q *a rational fraction,* $p \gtrless 0$, $q>0$; $\alpha_n (n=0, 1, 2, \cdots)$, *any positive or negative integer smaller in absolute value than a fixed arbitrary number* M, *but only a finite number of the* α_n *equal to* 0; *then*

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{a^{2^n}} \left(\frac{p}{q}\right)^n$$

is a transcendental number.

Professor Kempner states: "The condition that only a finite number of coefficients shall be zero $\cdots$ I have not been able to remove."

Now although the proof of the theorem appears essentially to depend not merely on the *croissance* of the denominators a^{2^n} but also on the particular character of the exponent 2^n of a, so that considerations based on the representation of numbers in the binary scale may be used, it nevertheless seems plausible that the restriction that only a finite number of coefficients shall be zero is dispensable. And, indeed, it is the purpose of this note to prove the theorem without this restriction; in other words, to prove the following theorem.

THEOREM. *The properly‡ infinite series*

$$f(x) = \sum_{n=0}^{\infty} \frac{\alpha_n}{a^{2^n}} x^n ,$$

where a is an integer greater than 1, *and* α_n *an integer less in absolute value than a fixed number* M, *is transcendental for rational* $x (\neq 0)$.

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† TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 477.

‡ That is, the terms are not all zero after a certain point.

PROOF. Let $x=p/q$, where $p(\neq 0)$ and $q(>0)$ are integers. Suppressing the terms of $f(x)$ in which $\alpha_n=0$ we may write

$$f\left(\frac{p}{q}\right)=\sum_{n=1}^{\infty}\frac{\alpha_{\sigma_n}}{\alpha^{2\sigma_n}}\,\frac{p^{\sigma_n}}{q^{\sigma_n}},$$

where $\alpha_{\sigma_n}\neq 0$ and $\sigma_n>\sigma_{n-1}(n=1,\ 2,\cdots)$. To prove that $f(p/q)$ is transcendental, we shall show that every polynomial

$$\phi(y)=A_0y^k+A_1y^{k-1}+\cdots+A_{k-1}y+A_k,$$

where $k\geqq 1$, $A_i(i=0,1,\cdots,k)$ is an integer, and $A_0\neq 0$, is different from 0 for $y=f(p/q)$. We distinguish the two possibilities: (a) for every n there are 2 consecutive σ_n's greater than n and differing by more than k; (b) after a certain point, the difference between 2 consecutive σ_n's is less than or equal to k.

Case (a). Let n be such that $\sigma_n>\sigma_{n-1}+k$. We suppose that the individual terms of the expansion of $\phi(f(p/q))$ are retained without cancellation of common factors of numerators and denominators. Because of the rapid increase of the denominators absolute convergence is assured. Out of the various denominators of $\phi(f(p/q))$, we single out the following three:

$$d_1=a^{(k-1)2^{\sigma_{n-1}}+2^{\sigma_{n-2}}}\,q^{(k-1)\sigma_{n-1}+\sigma_{n-2}},$$

$$d_2=\left(a^{2^{\sigma_{n-1}}}q^{\sigma_{n-1}}\right)^k,$$

$$d_3=a^{2\sigma_n-1}q^{\sigma_n}.$$

We have:

$$\frac{d_3}{d_2}\geqq\frac{a^{2^{\sigma_n}(1-k/2^{\sigma_n-\sigma_{n-1}})}}{q^{k\sigma_{n-1}}}$$

$$>\frac{a^{2^{\sigma_n}(1-k/2^k)}}{q^{k\sigma_{n-1}}}$$

$$\geqq\frac{a^{2^{\sigma_n-1}}}{q^{k\sigma_{n-1}}}$$

and

$$\frac{d_2}{d_1}>a^{2^{\sigma_{n-1}}-2^{\sigma_{n-2}}}\geqq a^{2^{\sigma_{n-1}}-2^{\sigma_{n}-1}}=a^{2^{\sigma_{n-1}}-1}.$$

From the expressions for d_1, d_2, and d_3 and their ratios, it appears that for n sufficiently large, d_1, d_2, and d_3 are consecutive if the denominators of $\phi(f(p/q))$ are arranged in ascending magnitude. The denominator d_2 occurs just once and with the numerator $n_2=A_0\alpha_{\sigma_{n-1}}^k p^{k\sigma_{n-1}}\neq 0$; and

$$\left|\frac{n_2}{d_2}\right| < \frac{A_0 M^k |p|^{k\sigma_{n-1}}}{d_1 a^{2^{\sigma_{n-1}}-1}} .$$

Hence

$$\frac{n_2}{d_2} = \frac{\epsilon_n}{d_1} ,$$

where $\epsilon_n \to 0$ as $n\to\infty$.

We shall now obtain an upper estimate of the absolute value of the sum s of those terms of $\phi(f(p/q))$ whose denominators are $\geqq d_3$. A typical term t of $\phi(f(p/q))$ has the form

$$A_{k-j}\frac{\alpha_{\sigma_{n_1}}\alpha_{\sigma_{n_2}}\cdots\alpha_{\sigma_{n_j}}p^{\sigma_{n_1}+\sigma_{n_2}+\cdots+\sigma_{n_j}}}{a^{2^{\sigma_{n_1}}+2^{\sigma_{n_2}}+\cdots+2^{\sigma_{n_j}}}q^{\sigma_{n_1}+\sigma_{n_2}+\cdots+\sigma_{n_j}}} ;$$

a multinomial factor m does not appear because, instead of regarding mt as appearing once, we regard t as appearing m times.

The number w_n of terms of $\phi(f(p/q))$ coming from the first n terms of $f(p/q)$ is $n^k+n^{k-1}+\cdots+n+1$ —we are supposing that the terms are kept without combination as they arise initially in the expansion of $\phi(f(p/q))$, and we include terms that may possibly be zero because some $A_j=0$; hence $w_n<n^{k+1}$. Hence surely the number of terms of $\phi(f(p/q))$ arising from the first n, but not from the first $(n-1)$ terms of $f(p/q)$ is less than n^{k+1}; moreover, any one such term is in absolute value less than $c|p|^{k\sigma_n}/a^{2^{\sigma_n}}$, where c represents the maximum absolute value of the numbers $A_{k-j}M^j$. Likewise, there are less than $(n+1)^{k+1}$ terms arising from the first $(n+1)$ terms, but not from the first n terms of $\phi(f(p/q))$; and each of these terms is in absolute value less than $c|p|^{k\sigma_{n+1}}/a^{2^{\sigma_{n+1}}}$. Therefore, since the terms of $\phi(f(p/q))$ with denominators $\geqq d_3$ involve the nth term or later terms of $f(p/q)$, we have

$$|s|<c\left[\frac{n^{k+1}|p|^{k\sigma_n}}{a^{2^{\sigma_n}}}+\frac{(n+1)^{k+1}|p|^{k\sigma_{n+1}}}{a^{2^{\sigma_{n+1}}}}+\cdots\right].$$

As may be seen through elementary considerations, the bracket is less than, say, double its first term, if n is sufficiently large. For sufficiently large n, we therefore have

$$|s|<\frac{2cn^{k+1}|p|^{k\sigma_n}}{a^{2^{\sigma_n}}}=\frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n}}{d_3}<\frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n+k\sigma_{n-1}}}{d_2a^{2^{\sigma_{n-1}}}}.$$

Since $n\leqq\sigma_n$, the relative magnitude of $a^{2^{\sigma_{n-1}}}$ renders the coefficient of $1/d_2$ infinitesimal for $n\rightarrow\infty$, so that

$$s=\frac{\epsilon_n'}{d_2},$$

where $\epsilon_n'\rightarrow 0$ as $n\rightarrow\infty$.

We may now see why $\phi(f(p/q))\neq 0$. For the sum of those terms of $\phi(f(p/q))$ whose denominators are $\leqq d_1$ is either 0 or at least $1/d_1$ in absolute value. In the former case,

$$\phi(f(p/q))=\frac{n_2}{d_2}+s=\frac{n_2+\epsilon_n'}{d_2}\neq 0,$$

since n_2 is integral and $\neq 0$. And in the latter case,

$$|\phi(f(p/q))|\geqq\frac{1}{d_1}-\frac{|\epsilon_n|}{d_1}-\frac{|\epsilon_n'|}{d_2}<\frac{1-|\epsilon_n|}{d_1}-\frac{|\epsilon_n'|}{d_1a^{2^{\sigma_{n-1}-1}}}\neq 0,$$

for sufficiently large n, because ϵ_n, ϵ_n', and $1/a^{2^{\sigma_{n-1}-1}}$ approach 0 as $n\rightarrow\infty$.

Case (b). As in case (a), we set

$$d_3=a^{2^{\sigma_n}}q^{\sigma_n}.$$

Let d_2 equal the largest denominator in $\phi(f(p/q))$ less than d_3; and d_1, the largest denominator less than d_2; it is understood that denominators are not excluded because their corresponding numerators in the expansion of $\phi(f(p/q))$ happen to be zero on account of the vanishing of one or more of the A's. The denominator d_2 is of the form $d_2=\prod_{\nu=1}^{h}a^{2^{\sigma_{j_\nu}}}q^{\sigma_{j_\nu}}$ where $h\leqq k$ and $\sigma_{j_\nu}<\sigma_n(\nu=1,2,\cdots,h)$. Moreover, $\sum_{\nu=1}^{h}2^{\sigma_{j_\nu}}<2^{\sigma_n}$. For since $h\geqq 2$, or else (for large n), as is particularly evident from the sequel, there are denominators between d_2 and d_3, $\prod_{\nu=1}^{h}q^{\sigma_{j_\nu}}\geqq q^{2\sigma_{n-k}}>q^{\sigma_n}$ for n sufficiently large, since (for large n)

$\sigma_{n-k} \geqq \sigma_n - k^2$. Consequently, if $\sum_{\nu=1}^{h} 2^{\sigma_{j_\nu}} \geqq 2^{\sigma_n}$, it follows that $d_2 > d_3$. Since, then, $\sum_{\nu=1}^{h} 2^{\sigma_{j_\nu}} < 2^{\sigma_n}$, we may conclude, if, as we supposed, there are to be no denominators between d_2 and d_3, that $h = k$ and that $\sigma_{n-1} \geqq \sigma_{j_\nu} \geqq \sigma_{n-k} (\nu = 1, 2, \cdots, k)$. (The inequality $\sum 2^{\sigma_{j_\nu}} < 2^{\sigma_n}$ guarantees, for n sufficiently large, that $\prod a^{2^{\sigma_{j_\nu}}} q^{\sigma_{j_\nu}} < d_3$ even though $\prod q^{\sigma_{j_\nu}} > q^{\sigma_n}$, as appears more clearly from the subsequent inequalities for d_3/d_2.) Hence we have, for large n,

$$\frac{d_3}{d_2} > \frac{a^{2^{\sigma_n}} q^{\sigma_n}}{a^{2^{\sigma_{n-1}} + 2^{\sigma_{n-2}} + \cdots + 2^{\sigma_{n-k}}} q^{k\sigma_{n-1}}} > \frac{a^{2^{\sigma_{n-k}}}}{q^{k\sigma_{n-1}}} .$$

From the definition of d_1, it follows that d_1, like d_2, is of the form $\prod_{\nu=1}^{k} a^{2^{\sigma_{j_\nu}}} q^{\sigma_{j_\nu}}$ and, indeed, the same quantities $a^{2^{\sigma_{j_\nu}}} q^{\sigma_{j_\nu}}$ occur, except that for one of these factors, say $a^{2^{\sigma_{j_\rho}}} q^{\sigma_{j_\rho}}$ of d_2, in which $\sigma_{j_\rho} = \min \sigma_{\lambda_\nu}$, is substituted the factor $a^{2^{\sigma_{j_\rho - 1}}} q^{\sigma_{j_\rho - 1}}$. Therefore

$$\frac{d_2}{d_1} \geqq \frac{a^{2^{\sigma_{n-k}}} q^{\sigma_{n-k}}}{a^{2^{\sigma_{n-k-1}}} q^{\sigma_{n-k-1}}} > a^{2^{\sigma_{n-k-1}}} .$$

The denominator d_2 is, except for the order of the factors $a^{2^{\sigma_{j_\nu}}} q^{\sigma_{j_\nu}}$, obtainable in just one way; if, therefore, n_2/d_2 is the sum of the terms of $\phi(f(p/q))$ with denominator d_2, we have $n_2 = A_0 \prod_{\nu=1}^{k} \alpha_{\sigma_{j_\nu}} p^{\sigma_{j_\nu}}$, where the σ_{j_ν} are the same as in d_2, and g is a multinomial coefficient in the expansion of a kth power. Hence $n_2 \neq 0$. Furthermore

$$\left| \frac{n_2}{d_2} \right| < \frac{A_0 g M^k |p|^{k\sigma_{n-1}}}{d_1 a^{2^{\sigma_{n-k-1}}}} \leqq \frac{1}{d_1} \cdot \frac{A_0 g M^k |p|^{k\sigma_{n-1}}}{a^{2^{\sigma_{n-1} - k^2}}}$$

for large n, since after a certain point, we have $\sigma_{\nu-1} \geqq \sigma_\nu - k$. Consequently

$$\frac{n_2}{d_2} = \frac{\eta_n}{d_1} ,$$

where $\eta_n \to 0$ as $n \to \infty$.

If, as in case (a), s represents the sum of the terms of $\phi(f(p/q))$ with denominators $\geqq d_3$, we have, as before, notwithstanding the fact that denominators $\geqq d_3$ may now arise from the first $(n-1)$ terms of $f(p/q)$,

$$|s| < \frac{2cn^{k+1}|p|^{k\sigma_n}}{a^{2\sigma_n}},$$

for large n. Hence

$$|s| < \frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n}}{d_3} < \frac{2cn^{k+1}|p|^{k\sigma_n}q^{\sigma_n+k\sigma_{n-1}}}{d_2a^{2\sigma_n-k}}.$$

Therefore

$$s = \frac{\eta_n'}{d_2},$$

where $\eta_n'\to 0$ as $n\to\infty$. From the fact that $n_2\neq 0$, the equations $n_2/d_2=\eta_n/d_1$ and $s=\eta_n'/d_2$, and the inequality for d_2/d_1, we conclude, as in case (a), that $\phi(f(p/q))\neq 0$.

As in the case of Professor Kempner's theorem, the exponent 2^n may be replaced by b^n, where b is an integer >2. It is also obvious that α_n need not be limited, but it suffices, for instance, to subject it—for large n—to being $\leqq Ml^n$, where M and l are positive constants, and $l<2$.

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