APPROXIMATE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS BY LEAST SQUARES*

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1. *Systems of First Order.* In many problems of mathematical physics and particularly in electrical circuit theory, it is of importance to find approximate solutions of a system of differential equations of the form

$$(1) \qquad \frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots, x_p), \qquad (i=1,2,\cdots,p).$$

Sometimes it may be shown by physical considerations that a system of type (1) which corresponds to some definite experimental fact, really possesses a periodic solution with a period equal to T. By a suitable change of variables, we may suppose that

$$(2) \qquad x_i = 0, \qquad (i=1,\cdots,p) \qquad \text{for} \qquad t=0, \qquad t=T;$$

and it then remains only to find the numerical solution of (1) and (2) with a given degree of approximation. We shall suppose first that the system (1) is linear with variable coefficients, that is to say, that

$$(3) \quad \frac{dx_i}{dt} - A_{i1}x_1 - A_{i2}x_2 - \cdots - A_{ip}x_p = F_i, \quad (i=1,2,\cdots,p),$$

where $A_{i1}, A_{i2}, \cdots, A_{ip}, F_i$ are functions of t. As in the method of least squares, we shall try to render stationary the integral

$$(4) \qquad \int_0^T \sum_{i=1}^{p} \left[\frac{dx_i}{dt} - \sum_{k=1}^{p} A_{ik}x_k - F_i \right]^2 dt$$
$$= \int_0^T \sum_{i=1}^{p} \left[L_i(x_i) - F_i \right]^2 dt$$

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by the means of a sequence of the type

$$x_i^{(n)} = \sum_{k=1}^{n} a_{ik}^{(n)} \varphi_k(t) , \tag{5}$$

where the functions $\varphi_k(t)$ satisfy* the boundary conditions (2). For what follows, we must suppose also that the results of the modern theory of summation (for example, that of L. Féjer, or that of D. Jackson) are applicable to the system $[\varphi_k(t)]$. If this be true, it is possible to indicate the degree of approximation not only of

$$|x_i - X_i^{(n)}| , \tag{6}$$

but also of

$$\left| \frac{dx_i}{dt} - \frac{dX_i^{(n)}}{dt} \right| , \tag{7}$$

where $X_i^{(n)} = \sum_{k=1}^{n} b_{ik}^{(n)} \varphi_k(t)$ is a partial sum of order n formed by means of Féjer's or Jackson's process, if A_{ik}, F_i are to be supposed to satisfy the well known Lipschitz condition.

If we suppose that the coefficients A_{ik}, F_i can be differentiated, the degrees of approximation of (6) and (7) are given by the recent invéstigations in the domain of the theory of functions of a real variable.† If we substitute (5) in (4) instead of x_i, and differentiate with respect to a_{ik}, it is easy to see that the conditions for a minimum take the form

$$\int_0^T \left\{ \left[L_i(x_i^{(n)}) - F_i \right] \left[\frac{d\varphi_k}{dt} - A_{ii}\varphi_k \right] + \sum_{r=1}^{r=p}{}_{(i)} \left[L_r(x_i^{(n)}) - F_r \right] A_{ri}\varphi_k \right\} dt = 0 . \tag{8}$$

* We may take, for example, $\varphi_k(t) = \sin kt$.

† See, e.g., de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Reélle*, Paris, Gauthier-Villars, 1919.

Multiplying the equations (8) respectively by $a_{ik}^{(n)} - b_{ik}^{(n)}$, adding them together, and assuming that the determinant of A_{ik} is symmetric, we obtain

$$\int_0^T \sum_{i=1}^{p} [L_i(x_i^{(n)}) - F_i][L_i(x_i^{(n)} - X_i^{(n)})]dt = 0 .$$

Evidently we have also

$$\int_0^T \sum_{i=1}^{p} [L_i(x_i) - F_i][L_i(x_i^{(n)} - X_i^{(n)})]dt = 0 ,$$

because x_i are the solutions of (2); therefore

$$\int_0^T \sum_{i=1}^{p} L_i(x_i - x_i^{(n)}) L_i(x_i^{(n)} - X_i^{(n)})dt = 0 .$$

But we have

$$x_i^{(n)} - X_i^{(n)} = x_i^{(n)} - x_i + x_i - X_i^{(n)} ;$$

hence

$$(9) \qquad \int_0^T \sum_{i=1}^{p} \left[L_i(x_i - x_i^{(n)}) \right]^2 dt$$

$$= \int_0^T \left\{ \sum_{i=1}^{p} L_i(x_i - x_i^{(n)}) L_i(x_i - X_i^{(n)}) \right\} dt .$$

If we utilise now the well known Bouniakowsky-Schwartz inequality, we find, from (9),

$$\int_0^T \sum_{i=1}^{p} [L_i(x_i - x_i^{(n)})]^2 dt \leqq \sum_{i=1}^{p} \sqrt{\int_0^T [L_i(x_i - x_i^{(n)})]^2 dt}$$

$$\times \sqrt{\int_0^T [L_i(x_i - X_i^{(n)})]^2 dt} .$$

Remembering what was said above about the approximation of x_i and dx_i/dt, respectively, by $X_i^{(n)}$ and $dX_i^{(n)}/dt$, we obtain from (9)

$$(10) \qquad \int_0^T [L_i(x_i - x_i^{(n)})]^2 dt < \epsilon_{in}, \qquad (i = 1, 2, \cdots, p),$$

where the order of magnitude of ϵ_{in} can be determined according to the restrictive conditions imposed on A_{ik} and F_i. From (10), by the means of the same Schwartz inequality, we find immediately

$$\left| \int_0^x L_i(x_i - \overset{(n)}{x_i})\,dt \quad \right| < \eta_{in}\,, \quad (i=1,2,\cdots,p),$$

where

$$\eta_{in} < \sqrt{\epsilon_{in}}\,.$$

Using the boundary conditions for $\varphi_k(t)$, we get the system of inequalities

$$\left| x_i - \overset{(n)}{x_i} + \int_0^x A_{ik}(x_k - \overset{(n)}{x_k})\,dt \right| < \eta_{in}.$$

The formulas for solving the system of integral equations of the second kind of the Volterra type enable us to state that

$$|x_i - \overset{(n)}{x_i}| < A\epsilon_{in}\,, \qquad (i=1,2,3,\cdots,p)$$

where the order of magnitude of ϵ_{in} can be indicated and A is a constant. Thus the following theorem is demonstrated.

THEOREM. *If the coefficients* A_{ik}, $F_i (i, k=1, 2, \cdots, p)$ *of the system* (3) *satisfy the Lipschitz conditions, and if the determinant formed from the* A_{ik} *is symmetric then the effective calculation of the solutions of the system* (3) *can be derived from the method based upon the minimizing of the integral* (4). *Such a method, which may be called the method of least squares, gives not only a convergent process, but also the possibility of calculating the solutions of* (2) *with a given degree of approximation.*

2. *Systems of Second Order.* If instead of the system (3) we have to integrate the system of the differential equations of the second order

$$(11) \qquad \frac{d^2x_i}{dt^2} - A_{i1}x_1 - A_{i2}x_2 - \cdots - A_{ip}x_p$$
$$= L_i(x_1x_2, \cdots, x_p) = F_i\,, \ (i=1,2,3,\cdots,p)\,,$$

with the boundary conditions $x_i(a)=x_i(b)=0$, then applying reasoning similar to that used before,* we can state that

$$\int_a^b [L_i(x_i-\overset{(n)}{x_i})]^2 dt < \epsilon_{in}\,, \qquad (i=1,2,\cdots,p), \tag{12}$$

where the order of smallness of ϵ_{in} can be fixed in advance corresponding to the supplementary restrictive conditions imposed on the coefficients A_{ik}, F_i. Then, if the determinant A_{ik} is symmetric we can start from the obvious identity.

$$\int_a^b \Big\{\sum_{i=1}^{p}\left[\frac{d(x_i-x_i^{(m)})}{dt}\right]^2 + [A_{11}(x_1-\overset{(m)}{x_1})^2$$

$$+2A_{12}(x_1-\overset{(m)}{x_1})(x_2-\overset{(m)}{x_2})+\cdots]\Big\}dt$$

$$=-\int_a^b \sum_{i=1}^{p} L_i(x_1-\overset{(m)}{x_1},\cdots,x_p-\overset{(m)}{x_p})(x_i-\overset{(m)}{x_i})dt\,.$$

If the quadratic form

$$A_{11}(x_1-\overset{(m)}{x_1})^2+2A_{12}(x_1-\overset{(m)}{x_1})(x_2-\overset{(m)}{x_2})+\cdots$$

is definitely positive, we easily obtain, by the use of the Bouniakowsky-Schwartz inequality,

$$\left|x_i-\overset{(m)}{x_i}\right|<\eta_m\,, \qquad (i=1,2,3,\cdots,p)$$

where the order of magnitude of η_m can be fixed in advance according to the order of ϵ_{in}.

In further communications, I shall state other applications of the method of least squares (and more generally of the method of least powers) which presents, it seems to me, a very powerful method for the approximate integration of differential equations of mathematical physics.

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* In the case of one equation of the second order it was first stated in my recent communication to the French Academy. See N. Kryloff, *Sur une méthode, basée sur le principe du minimum pour l'intégration approchée des équations différentielles*, COMPTES RENDUS, vol. 181 (1925).