A RAY OF NUMERICAL FUNCTIONS OF r ARGUMENTS*

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1. *Introduction.* Starting with any set Σ of elements we may combine them according to rules having the formal properties (commutativity, associativity, distributivity) of algebraic addition (A), multiplication (M), subtraction (S), and division (D), provided the resulting combinations can be assigned self-consistent interpretations. Assuming this to be the case for a given Σ, we may then investigate the properties of systems Σ' closed under one or more of A, M, D, S. It is not necessary to consider the properties of Σ' with respect to those of A, M, D, S that are omitted. There are thus conceivable precisely 15 systems Σ'. So far as systems Σ' consisting of numbers (rational integers, rational numbers, algebraic numbers, algebraic integers) are concerned, it appears that 4 of the possible 15 have been deemed of sufficient interest to receive technical names. These are as follows: AS, module (Modul); $AMDS$, field (Körper); AMS, ring (Ring); MD, ray (Strahl), the last being due to Fueter.†

The elements of Σ' need not be numbers to ensure interesting results, for example the algebra of classes and that of the relative product. Further significant theories have evolved from mere ova of Σ''s; thus the theory of partitions is the Σ' generated by a sort of parthenogenesis by A alone from given rational integers. Doubtless with the continued evolution of arithmetic the neglected 11 will also be born, baptized, and investigated. Should this indeed come to pass it is fortunate that algebra has but 4, not 4000, fundamental operations.

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† Introduced in his Dissertation; now current. Cf. *Synthetische Zahlentheorie* (Göschen's Lehrbücherei, 1921).

The ray isolates those properties of numbers and functions which are commonly called multiplicative.

The present note extends to $r>1$ arguments the ray devised in a former paper* to codify and extend the multiplicative properties of integers. The ray here relates to *sets* of r integers, $r \geqq 1$. As in the case $r=1$ it was pointed out that the elements taken as arguments of the functions concerned could be identified with the elements of any abelian group, so here there is easily obtainable the obvious generalization to functions whose arguments are taken from r abelian groups. Except in the special case when the numerical functions of $r>1$ arguments are products of r functions each of a single argument, there is no longer, as in the case $r=1$, a second isomorphism with algebraic functions in r variables. The properties of integers to which the case $r>1$ refers in general are genuine properties of a set of r arbitrary constant integers, and they constitute an extension to such sets of the multiplicative properties of a single integer (rational or algebraic; if the latter, resolved into ideal factors). An instance of a function of $r>1$ arguments which is not essentially the product of r functions of r single arguments is the G. C. D. of r integers.

The multiplication and division in the ray are specific operations having the abstract properties of MD, but not identical, in relation to the arguments, with the MD of common arithmetic.

2. *Multiplication.* Call $f=f(x_1, \cdots, x_r)$ a numerical function of $x_1, \cdots, x_r$ if f takes a single finite value when each x_i is an integer $\neq 0$, and if further $f(1, 1, \cdots, 1) \neq 0$.

Let $N_r=(n_1, n_2, \cdots, n_r)$ be an ordered set of r arbitrary constant integers each >0. Resolve n_i into t integral factors each $\geqq 1$,

$$n_i = n_{1i}\, n_{2i} \cdots n_{ti}, \qquad (i=1, 2, \cdots, r). \tag{1}$$

Let $f_j(x_1, \cdots, x_r)$, $(j=1, 2, \cdots, t)$ be numerical functions. The r-fold sum

* This BULLETIN, vol. 28 (1922), pp. 111–122.

(2) $\sum f_1(n_{11}, n_{12}, \cdots, n_{1r}) f_2(n_{21}, n_{22}, \cdots, n_{2r}) \cdots$
$f_t(n_{t1}, n_{t2}, \cdots, n_{tr}),$

in which $\sum$ refers to all $n_{ji}(j=1, \cdots, t;\ i=1, \cdots, r)$ as defined which satisfy (1), will be written

$$f_1 f_2 \cdots f_t \tag{3}$$

and will be called the product of $f_1, f_2, \cdots, f_r$ for the parameter N_r, or, when N_r is understood, simply the product.

The number of terms in (3) when distributed as a sum of terms each of the form (2) is

$$\prod_{j=1}^{t} \nu(n_j)\left\{\nu(n_j)-1\right\} \cdots \left\{\nu(n_j)-t+1\right\},$$

where $\nu(n)=$ the number of divisors of n.

From (1), (2) it follows that the multiplication just defined is commutative and associative.

The unit of this multiplication is the numerical function $u=u(x_1, \cdots, x_r)$, which vanishes except when $x_j=1(j=1, \cdots, r)$, in which case $u=1$. By (3), if f is any numerical function,

$$uf=f. \tag{4}$$

In (4), as in every equality between numerical functions, it is understood that the functions have the same parameter; thus the parameter of uf and of f is N_r.

The zero of multiplication is the numerical function w which vanishes for all parameters. Hence from (3),

$$wf=w. \tag{5}$$

3. *Division*. Precisely as in the case of functions of a single argument* it can be shown that for f any given

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*TÔHOKU MATHEMATICAL JOURNAL, vol. 17 (1920), pp. 221–231; or more readily by an immediate extension of the method of generators for the algebras called D, E in the TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 135–154. The generator of (3) is the ordinary product of t "r-fold Dirichlet series," such a series being

$$D_f \equiv \Sigma \frac{f(m_1, m_2, \cdots, m_r)}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r}},$$

where Σ refers to $m_1, m_2, \cdots, m_r$ each ranging from 1 to ∞, and $s_1, s_2, \cdots,$

numerical function $\neq w$ there exists a unique f', called the reciprocal of f, such that

$$ff' = u \ . \tag{6}$$

4. *The Ray* $\{f, g, h, \cdots\}$. From §§ 2, 3, it follows that the totality of numerical functions $f, g, h, \cdots$ of r arguments form a ray when multiplication and division are as above defined.

5. *Examples.* As a first example we state and prove an extension of Dedekind's inversion ($r=1$) to $r>1$. Let ζ be the numerical function whose value is 1 for all values of the parameter, and let ζ' be the reciprocal of ζ, so that $\zeta\zeta'=u$, the unit function. Then evidently

$$\zeta(n_1, n_2, \cdots, n_r) = \zeta(n_1)\zeta(n_2) \cdots \zeta(n_r) \ ,$$

and the explicit definition of ζ' is easily seen to be as follows: $\zeta'=0$ if any integer in the parameter is divisible by a square >1, otherwise $\zeta'=1$ or -1 according as the number of those integers in the parameter which are divisible by an odd number of primes is even or odd, and hence

$$\zeta'(n_1, n_2, \cdots, n_r) = \zeta'(n_1)\zeta'(n_2) \cdots \zeta'(n_r) \ .$$

Let f be any element of $\{f, g, h, \cdots\}$. Then by § 4, ζf is in the ray and, by what precedes,

$$F = \zeta f \qquad \text{implies} \qquad f = \zeta' F \ , \tag{7}$$

the extension in question.

As a second example, let a, b, f, g, h, k be elements of $\{f, g, h, \cdots\}$ between which there are the relations

$$af = bg \ , \qquad ah = bk \ . \tag{8}$$

Then, the indicated products being in the ray, we may eliminate a, b as follows,

$$\frac{af}{ah} = \frac{bg}{bk} \ , \qquad \frac{f}{h} = \frac{g}{k} \ ;$$

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s_r are independent ordinary parameters. The generator of u is $D_u=1$; that of f' in (6) is $1/D_f$. The generator of (3) is ΠD_f, where Π extends to $f_1, f_2, f_3, \cdots, f_i$. The condition $f(1, 1, \cdots, 1) \neq 0$ imposed in the definition in §2 is necessary to make the division of §3 always possible.

that is, (8) implies

$$fk = gh \ . \tag{9}$$

It may be of interest to state (8), (9) in ordinary notation: the equations

$$\sum a(n_{11}, \cdots, n_{1r})f(n_{21}, \cdots, n_{2r}) \\ = \sum b(n_{11}, \cdots, n_{1r})g(n_{21}, \cdots, n_{2r}),$$

$$\sum a(n_{11}, \cdots, n_{1r})h(n_{21}, \cdots, n_{2r}) \\ = \sum b(n_{11}, \cdots, n_{1r})g(n_{21}, \cdots, n_{2r}),$$

in which $\sum$ refers for $n_1, \cdots, n_r$ fixed to all positive integer solutions of

$$n_i = n_{1i}n_{2i} \qquad (i = 1, 2, \cdots, r) \ ,$$

together imply, for the same $\sum$,

$$\sum f(n_{11}, \cdots, n_{1r})k(n_{21}, \cdots, n_{2r}) \\ = \sum g(n_{11}, \cdots, n_{1r})h(n_{21}, \cdots, n_{2r}) \ .$$

For $r = 1$ this theorem is well known (due to Liouville).

With certain precautions and modifications (particularly as to the definition of zero elements) it is possible to construct for $f, g, h, \cdots$ a theory of each of the 15 possible types mentioned in the introduction. The ray however seems to be the only natural sequent to existing properties of numerical functions in the case $r = 1$.

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