THE APRIL MEETING IN CHICAGO

The twenty-fifth Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 2 and 3, 1926. The attendance numbered around one hundred, including the following fifty-nine members of the Society:

R. W. Babcock, Barnard, Basoco, W. S. Beckwith, Bibb, O. E. Brown, C. C. Camp, Curtiss, H. T. Davis, Dickson, Dostal, Dowling, Dresden, Ettlinger, Peter Field, C. A. Garabedian, Garver, Griffiths, Hazlett, Mildred Hunt, Dunham Jackson, R. L. Jackson, Krathwohl, E. P. Lane, McFarlan, McGavock, W. D. MacMillan, March, Marquis, William Marshall, Merriman, Mickelson, C. N. Moore, E. H. Moore, E. J. Moulton, F. R. Moulton, Nyswander, Olson, Phalen, Plant, Pounder, R. G. D. Richardson, Rider, H. L. Rietz, Rowland, Schottenfels, Shohat, Skinner, Slaught, Stouffer, E. L. Thompson, Van Vleck, Wait, L. E. Ward, R. L. Wilder, Williamson, F. E. Wood, Wyant, J. W. A. Young.

The Council announced the election to sustaining membership of the New York Edison Company.

The following sixteen persons were elected to ordinary membership:

Professor Charles Spencer Allen, Muhlenberg College;
Professor Charles Anthony Barnhart, State University of New Mexico;
Professor Talmon Bell, Sterling College;
Miss Dorothy Frances Briggs, Northeast Junior High School, Kansas City;
Mr. René Albrecht Carrié, College of the City of New York;
Professor Edward V. Casserly, Epiphany College;
Professor Rufus Crane, Ohio Wesleyan University;
Mr. Sven Bertil Hansell, New York Telephone Company;
Professor Francis M. Hartmann, Dean of Schools of Engineering, Cooper Union;
Mr. Svend E. Johannesen, General Electric Company;
Miss Marie Litzinger, Mount Holyoke College;
Professor Henry Howard Marvin, University of Nebraska;
Mr. Jerome Sidney Meyer, New York City;
Professor Charles Madison Sarratt, Vanderbilt University;
Mr. Paul A. Smith, Princeton University;
Professor Frederick Winchell Sparks, Louisiana State Normal College;

The secretary announced that Mr. P. L. Srivastava, of Oxford, had accepted membership under the reciprocity agreement with the London Mathematical Society.

It was announced that the following persons had been appointed to represent the Society: at the inauguration of President William Bennett Bizzell of the University of Oklahoma on February 4–5, Professor S. W. Reaves; at the fiftieth anniversary of the Société Scientifique de Bruxelles on April 12–13, President G. D. Birkhoff; at the twenty-fifth anniversary of James Millikin University on April 29, Professor J. B. Shaw; on the American Year Book, Professor Tomlinson Fort; on the Subcommittee on Mathematical Signs and Symbols of the American Engineering Standards Committee, Professor E. V. Huntington.

The invitations to Professors E. T. Bell and A. Pell-Wheeler to give Colloquium Lectures at the 1927 Summer Meeting in Madison have been accepted, their titles being, respectively, *Algebraic arithmetic*, and *The theory of quadratic forms in infinitely many variables, and applications.*

A committee to nominate trustees, officers, and members of the Council was appointed, consisting of E. R. Hedrick (chairman), G. A. Bliss, E. W. Chittenden, G. C. Evans, and H. H. Mitchell. A Committee on Arrangements for the Annual Meeting of 1926 was appointed to consist of Professors E. S. Crawley (chairman), G. H. Hallett, G. A. Harter, L. W. Reid, and the Secretary.

A statement from the Committee on Colloquia to be printed in the BULLETIN was authorized, and has since been printed (this BULLETIN, vol. 32, p. 100).

The symposium address, delivered on Friday afternoon by Professor Arnold Dresden, and entitled *Recent work in the calculus of variations*, will appear in full in an early issue of this BULLETIN. Titles and abstracts of the other papers read at this meeting follow below. Professor E. H. Moore's paper on generalized Hellinger integrals was given by invitation of the Assistant Secretary. On Friday morning the Society met in two sections; the papers numbered 1–10 were read before Section A (Point Sets and Geometry), Ex-President E. B. Van Vleck presiding, and those numbered 11–22 before Section B (Algebra and Analysis), Vice-President C. N. Moore presiding. Professor Moore presided also on Friday afternoon and Saturday morning, relieved during the second half of the Saturday morning session by Professor E. B. Skinner.

The papers by Gehman, Graves, Mathews, R. L. Moore, Nassau, H. L. Smith, Stouffer, and Williamson were read by title. Mr. R. G. Smith was introduced by Professor Stouffer, Mr. Whyburn by Professor R. L. Moore, and Mr. Williamson by Professor Dickson.

1. Mr. G. T. Whyburn: *Concerning point sets which can be made connected by the addition of a simple continuous arc.*

The author makes a study of conditions which a point set must satisfy in order that an arc may be drawn in such a way as to contain at least one point of every maximal connected subset of that point set. In particular it is proved that a closed and bounded point set M can be made connected by the addition of some simple continuous arc if and only if for every continuum K of M which consists of more than one point there exists a positive number ϵ_K such that K is not the limiting set of any set of maximal connected subsets of M each of diameter less than ϵ_K. This theorem is modified and extended so as to apply to point sets which are not bounded and also to sets which are not closed. A necessary and sufficient condition is given in order that a point set should be a subset of an open curve.

2. Professor R. L. Wilder: *A connected and regular point set which has no subcontinuum.*

It has been shown by R. L. Moore (this BULLETIN, vol. 29 (1923), p. 438) that there exist, in the plane, connected and regular (connected im kleinen) point sets which contain no arc. The present paper establishes the more general fact that there exist, in the plane, connected and regular point sets which contain no continuum (i.e., which contain no closed and connected subsets).

3. Professor R. L. Wilder: *A characterization of continuous curves by a property of their open subsets.*

If A and B are two distinct points of a point set N, then A and B are *separated in N in the weak sense* provided N contains no connected subset which contains both A and B; A and B are *separated in N in the strong sense* if there exist two mutually separated sets whose sum is N and which contain A and B respectively. The following theorems are proved: (I) If M is a closed and bounded point set, and A and B are points of M which are separated in M in the weak sense, then A and B are separated in M in the strong sense; (II) Theorem I holds if M, instead of being bounded and closed, is regular; (III) In order that a bounded continuum M should be a continuous curve, it is necessary and sufficient that every two points which lie in an open subset N of M and are not separated in N in the strong sense be not separated in N in the weak sense.

4. Professor R. L. Wilder: *A property which characterizes simple continuous curves.*

In an earlier paper* the author has made use of sets $K(C_1, C_2)M$ to characterize a continuous curve M. In the present paper it is shown that in order that a continuum M shall be a simple continuous curve, it is necessary and sufficient that every set $K(C_1, C_2)M$ be an arc.

5. Professor R. L. Moore: *A characterization of a point set which contains no domain.*

It is shown in this paper that in order that the point set M shall contain no domain it is necessary and sufficient that for every two points A and B in M there exist a totally discontinuous subset of M which separates A from B in M.

6. Dr. H. M. Gehman: *Concerning end points of continuous curves and other continua.*

Several definitions of an end point of a continuum have been given using properties possessed by an end point of a straight line interval. Yoneyama† has given three definitions by means of properties which he calls (A), (B), and (C). R. L. Wilder‡ uses a property which we shall call (W), to define an end point of a continuous curve. We here discuss the following two additional properties: A point P of a continuum M is said to have property (W'), if for every subcontinuum K of M containing P, the point P is not a limit point of any maximal connected subset of $M-K$; it is said to have property (M), if it is not a cut point of any subcontinuum of M. In the case of continuous curves, we find that (C) implies (B), and (B) implies each of the others, while (A), (W), (W'), and (M) are equivalent. Therefore, for continuous curves, (A), (W'), or (M) may be used as an alternative to Wilder's definition of an end point. But in the case of continua in general, the only logical relations connecting these properties are the following: (C) implies (B); (B) implies (M); (W') implies (A); (A) implies (M); (A) and (C) together imply (W'). Property (W) is omitted from this discussion, since in general it is not applicable to continua that are not continuous curves.

7. Professor E. P. Lane: *The correspondence between the tangent plane of a surface and its point of contact.*

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* *A property which characterizes continuous curves*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 11 (1925), pp. 725–28.

† TÔHOKU MATHEMATICAL JOURNAL, vol. 17 (1920), p. 253.

‡ FUNDAMENTA MATHEMATICAE, vol. 7 (1925), p. 358.

In the literature of projective differential geometry, there are instances of the intuitive use of the correspondence between the tangent plane of an analytic surface and its point of contact. An analytic formulation of this correspondence is furnished in the present paper, and some applications of it are made. In particular, it is used to define a quartic Cremona transformation between points in the tangent plane of a surface and planes through the contact point.

8. Professor E. P. Lane: *Comparison of Wilczynski's, Green's, and Fubini's canonical systems of differential equations for a surface.*

Wilczynski's canonical equations for a surface referred to its asymptotic net are $y_{uu}+2by_v+y=0$, $y_{vv}+2a'y_u+gy=0$. Fubini's normal coordinates are solutions of the equations $y_{11}=\beta y_2+ny$, $y_{22}=\gamma y_1+\nu y$, in which subscripts indicate covariant differentiation with respect to the quadratic differential form whose vanishing gives the asymptotic net. The present author studies the connection between these two systems of equations. Moreover, Green's canonical equations for a surface referred to a conjugate net are $y_{uu}=ay_{vv}+by_u+cy_v+dy$, $y_{uv}=b'y_u+c'y_v+d'y$, where $b+2c'=0$, $2ab'-c-a_v=0$. The corresponding equations in covariant derivatives are obtained and studied. In particular, a means is furnished for distinguishing between properties of a conjugate net and properties of its sustaining surface.

9. Dr. R. M. Mathews: *Cubic curves and desmic surfaces.* Second paper.

The parametric specification of the points of a cubic curve and of the points of a desmic surface in terms of the same elliptic functions suggests an intimate relation between the two classes of figures. This is investigated, and the geometric significance of the absolute invariant of the elliptic function is found for the surface. Among the theorems proved we may note the following: *The locus of the vertices of projective cubic cones in a desmic complex consists of six conjugate desmic surfaces.*

10. Professor J. S. Turner: *On the irrationality of the trigonometric functions of $n\pi$, where n is rational.*

In this paper it is proved that the only positive acute angles of the form $n\pi$, where n is rational, which possess trigonometric functions of the form $\sqrt{m}$, where m is rational, are $\pi/3$, $\pi/4$, $\pi/6$.

11. Professor G. E. Wahlin: *The conductor of a ring in an algebraic number field.*

The system of linear equations expressing the fundamental system of a ring in an algebraic number field in terms of the fundamental system of the field has a determinant C known as the index of the ring. The conductor of the ring is a factor of C. By means of the elementary divisors of C, the author obtains properties of the conductor which enable him to determine integers of the field which have the prime power factors of the conductor as their greatest common divisor.

12. Professor E. B. Stouffer and Mr. R. G. Smith: *Some relations involving sums of determinants.*

The main theorem of this paper states that the sum of all the determinants of order n obtained by combining in a certain definite manner the rows of $p+1$ determinants of order n is equal to the sum of the determinants obtained by combining the columns in a similar manner. The proof depends merely upon the order of application of certain differential operators. The theorem has as special cases an extensive list of theorems proved by LePaige, Deruyts, Muir, and others.

13. Professor E. B. Stouffer: *On determinants expressed in terms of their principal minors.*

In an earlier paper (TRANSACTIONS OF THIS SOCIETY, 1924) the author obtained several expressions for the general determinant in terms of its principal minors. The simplest case was for the determinant of order four. In the present paper it is shown that every such expression for the determinant of order four gives immediately an expression for the determinant of any higher order in terms of its principal minors. Moreover, the expressions for the general determinant of any order each involve only fourteen minors. The proof depends upon the so-called law of extensible minors.

14. Professor J. A. Shohat: *Some new applications of Tchebycheff polynomials.*

The author applies a method developed in a previous paper (*On a general formula,* read before the Society at Ithaca, September 11, 1925) to the following problem. Consider all polynomials $G(x)=\Sigma_{i=0}^{n} g_i x^i$ of degree $\leqq n$, satisfying the condition $\int_a^b q(x)G^2(x)dx \leqq M((a,b)$ finite or infinite, $q(x)\geqq 0$ on (a,b), $M>0)$; find the maximum, actually attained, of $|\Sigma_{i=0}^{n}\alpha_i g_i|$, where the α_i are arbitrarily given real constants. A complete solution is obtained in terms of Tchebycheff polynomials. Specifying $q(x)$ and the α_i, we get numerous results concerning polynomials in general and Tchebycheff polynomials in particular.

15. Professor H. J. Ettlinger: *On multiple iterated integrals.*

The author discusses the properties of m-fold multiple iterated integrals for which the multiple integral itself may not exist. A sufficient condition that the $m!$ iterated integrals of a function of m variables exist and be equal is that the integrand be bounded and be integrable in each variable for all fixed values of the others, save for a discrete set. A very general theorem of summation is found for evaluating m-fold iterated sums in terms of m-fold iterated integrals. This theorem is of importance in passing from the algebraic system to the Fredholm integral equation for a very general type of kernel in which integrability in each variable is required for fixed values of the other save for a discrete set. These conditions admit of a solution in which iterated integrals of the above type enter.

16. Professor P. R. Rider: *A figuratrix for double integrals.*

In a paper presented to the Society December 26, 1924, the author called attention to a number of interesting properties of the figuratrix in the calculus of variations. In the present paper he discusses a figuratrix (a surface) for double integrals, incidentally developing the Euler-Lagrange equations for a new form of the problem of double integrals, and shows that a certain necessary condition obtained by Kobb implies that the total curvature of the figuratrix must be positive.

17. Professor H. A. Simmons: *The first and second variations of a double integral in the case of variable limits.*

In this paper a formula is developed for the derivative of a double integral with respect to a parameter which appears in both the integrand and the limits of the integral. Application of this formula is made to calculate the second variation for a problem of the calculus of variations in which the minimum of a double integral is sought in a class of surfaces whose boundaries are not on a fixed curve but are allowed to lie on a fixed surface. This seems to be the first time that the second variation of this type has been calculated. The calculation has been regarded in the literature as a difficult one. From the second variation so attained a boundary value problem is formulated which is closely associated with the Jacobi condition of the problem of the calculus of variations of first dimension, and a formulation of that condition in terms of the smallest characteristic parameter value of the boundary problem is given.

18. Dr. L. E. Ward: *Functions expansible in two particular types of series.*

This paper deals with the possibility of expanding an arbitrary function in each of two series, one whose terms are characteristic functions of the differential system $u'''+\rho^3 u=0$, $u(0)=u'(0)=u(\pi)=0$, and another whose terms are characteristic functions of $v'''+\rho^3 v=0$, $v(\pi)=v'(\pi)=v(0)=0$. The first of these series converges in a triangle centered at $x=0$, and the other in a triangle centered at $x=\pi$. It is found that if both series are to converge, the function must be an elliptic function of special type; and that the two triangles of convergence will have a region of the plane in common if the poles of the elliptic function are situated properly.

19. Professor H. T. Davis: *Solution of the Laplace differential equation of infinite order by symbolic operators.*

E. Hilb* has studied the question of solving the non-homogeneous differential equation of infinite order of Laplace type by reducing it to an infinite system of linear equations in an infinite number of unknowns and applying to this system the methods of Hilbert and Schmidt. In this paper the same problem is approached from the point of view of symbolic opera-

* MATHEMATISCHE ANNALEN, vols. 82 and 84.

tors. It is shown how a solution can be found for Bourlet's generatrix equation and that the operator thus obtained is capable of expansion into a factorial series. The convergence properties of the solution are found to depend upon the behavior of an operator of the form $A_n = D(1-D)$ $(1-D/2)\cdots(1-D/(n-1))$. A solution due to H. Mellin* of the equation $u(x+1)-xu(x)=f(x)$ is obtained as a special case.

20. Dr. C. C. Camp: *A method for accelerating the convergence in the process of iteration.*

The popularity of the method of successive approximations has been imperiled by the extreme slowness of the convergence in many cases. The present paper overcomes the difficulty by furnishing a powerful method based on the same kind of analysis as Newton's but better adapted to the process of iteration. In large classes of problems the method is more powerful than Newton's. Certain modifications simplify the arithmetic and diminish further the number of approximations needed. The unmodified method consists of one iteration $f_1(x_k)=f_2(x_{k-1})$ and one application of the formula $\bar{x}_k = x_{k-1}+(x_k-x_{k-1})/(1-m)$, where $m=f_2'(x_{k-1})/f_1'(x_k)$. If $|m| \geqq 1$, the iterative process fails. We therefore assume $|m|<1$, which means geometrically that certain tangents must intersect. It is safer to take an interval so small that neither f will have an inflexion therein. The author determines the rapidity of convergence of ordinary iteration, and compares that for his method with that for the method of Newton. The method has been extended to the case of two equations in two unknowns, and to the case of complex roots of a single equation. It is capable of extension to n equations.

21. Professor L. E. Dickson: *Diophantine equations and quaternary quadratic forms.*

Numerous results are obtained by means of generalized quaternions having $\beta=1$ or ± 3 (*Algebras and their Arithmetics*, pp. 187, 193–94). Let $N(q)$ denote the quadratic form which is the norm of an integral element q. All integral solutions of $N(q)=x_1 \cdots x_k$ are obtained from $V=d_1\cdots d_k$, $x_i=N(d_i)$, if we employ arbitrary integral elements for $d_1, \cdots, d_k$. By multiplying q by suitably chosen elements we obtain an element with integral coordinates (coefficients x, y, z, w of $1, i, E, iE$). Hence the preceding theorem holds also when $N(q)$ denotes $x^2+y^2+\beta(z^2+w^2)$ and when the d_i have integral coordinates. There is found the number of all representations and the proper representations of any integer by each of these six quadratic forms. Corresponding results are obtained by other methods when $\beta=7$. A like formula gives all integral solutions when q is replaced by an n-rowed square matrix and $N(q)$ by its determinant. The subject is developed in detail in the German edition of the book cited which will be published this summer by Orell Füssli Verlag in Zurich.

22. Mr. John Williamson: *Division algebras connected with a nonabelian group of three generators.*

* ACTA MATHEMATICA, vol. 15 (1891), pp. 317–384.

It has been shown by L. E. Dickson (TRANSACTIONS OF THIS SOCIETY, April, 1926) that with every solvable group G there is connected a system of division algebras. The present paper treats in detail the two cases in which G is generated by three generators Θ_1, Θ_q, and Θ_p, of respective orders q, Q, and P, respectively, where Θ_p transforms Θ_1 and Θ_q into their respective inverses, and in the first case Θ_1 and Θ_q are commutative but in the second Θ_q transforms Θ_1 into its inverse. The conditions that these algebras be associative are obtained, and examples of groups of the required type are exhibited.

23. Professor H. L. Smith: *On relative content and Green's lemma.*

Definitions and elementary properties of content and of absolute content of planar sets relative to a function of sets not necessarily additive are given, and Green's lemma is obtained in the form of an equation between the line integral $\int P(x,y)dy$ around a simple closed curve and the content of that curve relative to a function of sets derived from $P(x,y)$. Certain special cases are exhibited in which the boundary curve need not be rectifiable.

24. Professor H. L. Smith: *The existence of Minkowski's linear measure for a simple curve.* Preliminary report.

The author shows that the length of a simple curve and its linear measure in the sense of Minkowski both exist and are equal provided either exists.

25. Professor Dunham Jackson: *Note on a problem in approximation with auxiliary conditions.*

This paper appears in the present issue of this BULLETIN.

26. Professor R. W. Babcock: *On thermal convection between plates.*

In convection between infinite horizontal parallel plane plates with space-periodic impressed boundary temperatures of opposite phase, the temperature distribution within the liquid assumes different types as the distance between the plates is changed. The variations from the temperature due to conduction alone are localized near the boundary surfaces, except where these surfaces are sufficiently close together to produce a considerable temperature gradient.

27. Mr. G. M. Merriman: *A set of necessary and sufficient conditions for the Cesàro summability of double series, with application to the double Fourier series.*

The present paper fills a long felt gap in the theory of summability of double series, by deriving a set of conditions necessary and sufficient that a double series be summable, after Cesàro, with respect to some pair of means or other; heretofore conditions for summability have been given that are either necessary or sufficient, but never both. These results are then applied to the case of the double Fourier series of a function that is integrable (Lebesgue), in which case the conditions are particularly simple and useful. Incidental to this application, there is derived a necessary and sufficient condition for the convergence of a double Fourier series of a certain type. The main results are generalizations to two variables of conditions for the Cesàro summability of simple series obtained in 1923 by Hardy and Littlewood.

28. Professor C. N. Moore: *On the construction of Fourier's series converging to functions with given singularities.*

Relatively little is known concerning the relationship between the singularities of a function of a real variable and the Fourier coefficients corresponding to that function. In this paper, a study of this relationship is begun by developing methods for the construction of Fourier's series whose sum-functions have singularities at certain preassigned point sets.

29. Professor E. H. Moore: *Introduction to a theory of generalized Hellinger integrals.*

This paper will appear in full in an early issue of this BULLETIN.

30. Dr. L. M. Graves: *Some theorems concerning measurable functions.*

The first two theorems concern the measurability of functions of measurable functions. The third theorem concerns the summability of the function $\int_a^x f(x,y)dy$, where f is a summable function in the plane.

31. Professor J. J. Nassau: *Some extensions of the generalized Kronecker symbol.*

This paper is concerned with the Kronecker symbol as generalized by Murnaghan (AMERICAN MATHEMATICAL MONTHLY, vol. 32). Its aim is to exhibit some possibilities of this notation in expressing certain functions, and in deriving conclusions. The expression for a permanent is given by

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}^2 a_1^{\alpha_1} a_2^{\alpha_1} \cdots a_n^{\alpha_n}.$$

A number of theorems relating to permanents and determinants and their minors have been developed. Extension of the symbol to determinants with multiple suffixes has been indicated. The symbol is well adapted to expressing symmetric functions. The expression for $\Sigma a_1^{p} a_2^{p_2} \cdots a_r^{p_r}$ of n letters is given by

$$\frac{1}{(\bar{n}-r)!}\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix}^2 a_{\alpha_1}^{p_1} a_{\alpha_2}^{p_2} \cdots a_{\alpha_r}^{p_r}.$$

Lastly a modification is introduced into the original definition to make possible the representation of some analytical functions recently considered by Macmahon (CAMBRIDGE PHILOSOPHICAL SOCIETY TRANSACTIONS, vol. 23).

R. G. D. RICHARDSON,
Secretary.