ON A CERTAIN FUNCTIONAL CONDITION*

BY J. P. BALLANTINE

A mean, x_3, between two numbers, x_1 and x_2, is obtained by use of the formula

$$p_1 f(x_1) + p_2 f(x_2) = (p_1 + p_2) f(x_3), \tag{1}$$

where p_1 and p_2 are arbitrary weights and $f(x)$ is any of several functions. If the function chosen is x itself, the resulting mean is the arithmetic mean; if $1/x$, the harmonic mean; if $\log x$, the geometric mean; if x^2 the mean-square. Since this terminology affords no hint for a generalization, we may as well call the general mean given by (1) the f-mean.

We have named all the means in common use. Why not, by use of the above generalization, extend the notion to, say, the "sine-mean"? This will probably not be done, principally because the proposed mean does not possess a certain useful property which is characteristic of all the ordinary means. This property is simply that multiplication of x_1 and x_2 by any constant results in the multiplication of x_3 by the same constant.

Let us study this property, and see what conditions it imposes on the function whose mean possesses it. We will replace (1) by the symmetrical equations

$$p_1 f(x_1) + p_2 f(x_2) + p_3 f(x_3) = 0 \tag{2}$$
$$p_1 + p_2 + p_3 = 0 .$$

The desired property is expressed by

$$p_1 f(ax_1) + p_2 f(ax_2) + p_3 f(ax_3) = 0 , \tag{3}$$

where $p_1, p_2, p_3, x_1, x_2, x_3$ are any set of numbers satisfying (2), and a is any constant. We desire to find all functions $f(x)$ which

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are such that, for every value of a, every set of numbers p_1, p_2, p_3, x_1, x_2, x_3 satisfying (2) also satisfies (3).

It will be no very regrettable further restriction on $f(x)$ to require that its derivative exist, at least in some interval. Assuming x_1, x_2, and x_3 to lie in that interval, we will differentiate (3) with respect to a and set $a=1$; this gives

$$p_1x_1f'(x_1)+p_2x_2f'(x_2)+p_3x_3f'(x_3)=0\ . \tag{4}$$

Though in the first paragraph we supposed, in (1), x_1, x_2, p_1, and p_2 given and x_3 to be found, it is clear that after x_3 is found what we have is simply a set of numbers x_1, x_2, x_3, p_1, p_2, and p_3 satisfying (2). There is also perfect symmetry among the subscripts 1, 2, and 3, so that by permuting the weights p_1, p_2, and p_3 any x is a mean between the other two. In obtaining the set of six quantities satisfying (2), it makes no difference which are taken arbitrarily and which are computed. It can easily be shown that, if we take x_1, x_2, and x_3 arbitrarily, there exists a set of numbers p_1, p_2, and p_3, not all zero, satisfying equations (2). For, if $f(x_1)=f(x_2)=f(x_3)$, any set satisfying the second equation of (2) satisfies the first, while if this is not the case, the two-row minors of the matrix of coefficients of p_1, p_2, p_3, in (2), will not all vanish, and hence a solution p_1, p_2, p_3, not all zero, exists.

Let us suppose, then, that x_1, x_2, and x_3 are taken arbitrarily. We have shown that there exists a set p_1, p_2, p_3, not all zero, which together with the assigned x_1, x_2, x_3, satisfy (2). Therefore, by the condition imposed on $f(x)$, this same set of six quantities satisfies (3) and hence (4). A necessary condition for the existence of p_1, p_2, p_3, not all zero, satisfying (2) and (4) is

$$\begin{vmatrix} 1 & 1 & 1 \\ f(x_1) & f(x_2) & f(x_3) \\ x_1f'(x_1) & x_2f'(x_2) & x_3f'(x_3) \end{vmatrix} = 0\,. \tag{5}$$

If, now, x_1 and x_2 are fixed, and x_3 is taken as the variable x, (5) reduces to a differential equation of the form

$$Axf'(x)+Bf(x)+C=0\,. \tag{6}$$

If $A=0$, $f(x_1)=f(x_2)$, whence, from (2), $f(x_3)=f(x_1)$ for all values of x_3, i.e., $f(x)$ is a constant, a trivial case.

If $A\neq 0$, and $B\neq 0$, the solution of (6) is

$$f(x)=\frac{-Kx^n-C}{B} ,$$

where K is arbitrary, and $n=-B/A$. If $n=1$, the above function produces the arithmetic mean; if $n=-1$, the harmonic mean; and if $n=2$, the mean-square. The values of K, B, and C (except the effect of B on n) have no effect on the corresponding f-mean.

If $A\neq 0$ and $B=0$, the solution is

$$f(x)=\frac{-C\log x}{A}+K ,$$

where K is arbitrary. For all values of C, A, and K, the corresponding f-mean is the geometric mean.

Therefore, excluding additive and multiplicative constants, which obviously have no effect on the corresponding f-mean, the only functions having the required property are x^n and $\log x$. Direct verification shows that both of these functions have the required property.

COLUMBIA UNIVERSITY