NOTE ON THE SECOND LAW OF THE MEAN FOR INTEGRALS*

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The second law of the mean may be stated as follows:

Given $f(x)$ a monotonic function and $\varphi(x)$ an integrable function in the interval $a \leqq x \leqq b$. Then there always exists a value of x, $x=\xi$, of the interval such that

$$(1) \qquad \int_a^b f(x)\varphi(x)\,dx = f(a)\int_a^\xi \varphi(x)\,dx + f(b)\int_\xi^b \varphi(x)\,dx\,.$$

It is the purpose of this note to prove that ξ *may always be chosen interior to the interval.*

For convenience of proof we may assume without loss of generality that $f(x)$ is defined at every point of the interval, that it is a monotonic increasing function, and that two values of x, $x=\eta$, $x=\epsilon$, $\eta<\epsilon$, exist interior to the interval such that $f(a)<f(\eta)<f(\epsilon)<f(b)$. We shall then assume that ξ equal to one of the end points, say a, is a known possible choice, and we shall prove that in this case a choice of ξ interior to the interval is also possible. We then have

$$(2) \qquad \int_a^b f(x)\varphi(x)\,dx = f(b)\int^b \varphi(x)\,dx\,.$$

Since $\varphi(x)$ is integrable, we may set

$$g(x) = \int_a^x \varphi(x)\,dx\,,$$

and $g(x)$ is then continuous. Integrating by parts† the left hand side of (2), we obtain

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† Hobson, *The Theory of Functions of a Real Variable*, 2d ed., vol. 1 (1921), pp. 607-608.

$$\int_a^b f(x)\varphi(x)\,dx = f(b)\int^b \varphi(x)\,dx - \int_a^b g(x)df(x)\,.$$

Hence the necessary and sufficient condition for the existence of relation (2) is

$$\int_a^b g(x)\,df(x) = 0\,. \tag{3}$$

By subtracting (1) from (2), we obtain for the necessary and sufficient condition for the existence of a ξ interior to the interval

$$\int_a^\xi \varphi(x)\,dx = 0\,,$$

which may be written $g(\xi) = 0$. If we then deny the existence of such a point ξ we have $g(x) \neq 0$ in the interior of the interval. Since $g(x)$ is continuous it is then of constant sign in the interval and we may assume it positive. Then

$$\int_a^b g(x)\,df(x) \geqq \int_\eta^\epsilon g(x)\,df(x)\,.$$

But in this latter interval $g(x)$ is everywhere positive and so has a lower limit $G > 0$. Hence

$$\int_\eta^\epsilon g(x)\,df(x) \geqq G\int_\eta^\epsilon df = G[f(\epsilon) - f(\eta)] > 0\,.$$

So we are led to a contradiction with (3), and it follows that the existence of the required interior point ξ is proved.*

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* Since $\int_a^b g(x)df(x)$ is a Stieltjes integral the equation (3) holds when for $f(b)$ we put any number $B \geqq f(b-0)$ and consequently the discussion of this note holds for the more general form of the second law of the mean of Du Bois-Reymond, Pringsheim, and de la Vallée Poussin. See Pringsheim, MÜNCHENER SITZUNGSBERICHTE, vol. 30 (1900), p. 211; and de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 2d ed., vol. 2 (1912), pp. 54–55.