THE FIRST CARUS MONOGRAPH

Calculus of Variations. By Gilbert Ames Bliss. Carus Mathematical Monographs, No. 1. Published for the Mathematical Association of America by the Open Court Publishing Company, Chicago, 1925. 13+189 pp.

One of the important concepts introduced into mathematical thought by E. H. Moore is that of the "Extensional Attainability" of properties (see New Haven Colloquium Lectures, p. 53). This concept, apart from its technical significance in the theory of properties of classes, admits of another interpretation of perhaps broader applicability, one suggestive of conquest. So, an infant reaching out for its playthings might be said to be experimenting with the extensional attainability of satisfaction for its desires; explorers illustrate the extensional attainability of man's control over the globe; other instances will occur to the reader.

The Carus Mathematical Monographs, of which Professor Bliss's book is the first, are intended "to contribute to the dissemination of mathematical knowledge by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in pure and applied mathematics," "in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields, and especially to the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises." Is this not an exhibition of faith in the extensional attainability of a mathematically informed public? It certainly is most fitting that this series of monographs should have been conceived by a Chicago group and that its first number should come from the pen of one of the members of the Department of Mathematics at the University of Chicago.

To Mrs. Mary Hegeler Carus and to her son Dr. Edward Carus belongs the honor of having recognized the importance of such an undertaking and of having provided the necessary means. Through the publication of these books, the Open Court Publishing Company continues its fine service to mathematical education in this country.

To Professor H. E. Slaught belongs the credit for the inception of the idea which gave rise to the monographs and for having solicitously guided it to successful realization. This series of books will forever be a reminder of his farsighted and intelligent devotion to the cause of mathematical education and to his skill in leading it on into new and significant fields of conquest.

In how far the wider dissemination of knowledge contributes to the enlargement of its domain, not many would venture to say. And, whether mathematics is better served by making results long familiar to the specialists accessible to a wider group than by the publication of new results, is, or should be, a rather futile question. Neither of these tasks should be allowed to be neglected. We must constantly labor to enlarge the foundations if they are to support the ceaselessly expanding superstructure, without, however, using up so much material in the process that the superstructure must suffer.

The simile is manifestly inadequate because it does not allow for the human elements involved. Diffusion of ideas among a larger group is likely to generate curiosity, to lead to the discovery of hitherto unsuspected connections, to suggest new ideas. There is no doubt that a much more healthy development may be expected if broader connecting avenues are laid out between the fields of pure and applied mathematics, if the engineers and the physicists became acquainted with some of the developments and some of the results that have been secured from ideas that perhaps were first brought to light in their fields of knowledge. If the Carus monographs contribute to a removal of the barriers which keep modern mathematical science isolated, they will fully justify the faith of the founders and the hopes of its promoters. May the series prove to be a multiple series stretching out its arms in many directions, so as to attain the widest possible extension.

In accordance with the general aim set for the series, the volume now under review has the purpose of bringing the general methods of the calculus of variations within the reach of a larger public than can be expected to master all the requirements which a systematic study of the subject would require. And to the reviewer it seems that the book succeeds admirably in this purpose. Whether this judgment is correct can only be estimated by the future historian.

The process is inductive. In an introductory chapter of sixteen pages, the calculus of variations is presented to the reader in a semihistorical way, chiefly by means of an analogy with the problem of finding the maxima and minima of functions, and through some of its illustrative problems. Three of these, all belonging to the "simplest problem of the calculus of variations," viz. the problems of the shortest distance, of the brachistochrone, and of the surface of revolution of minimum area, form the subject matter of chapters II, III, and IV, respectively, a total of 111 pages. The fifth chapter treats the minimizing of the integral $\int f(x, y, y')\,dx$. Then follow a list of references, notes, and index.

"The author assumes," so the warning on the jacket reads, "that the reader has an acquaintance with the elementary principles of the Differential and Integral Calculus." Even with this assumption as to the amount of preparation of his readers, the author must have been in doubt many times as to whether or not to presuppose knowledge of a particular fact. In some instances he apparently concluded that a reminder was the only thing necessary. As such we have to consider for instance the clause on page 21 concerning the derivative of an integral with respect to its upper limit, and the brief paragraph on the cycloid on page 52. Would not references to fuller treatment of such questions be useful in these places?

Each of the problems to which separate chapters are devoted receives a more complete treatment than they do when used as examples following the development of the general theory, a treatment moreover which utilizes to the full the more recent work done in connection with the classical theory. In preparation for the derivation of the Euler equation in the shortest distance problem, the lemma, usually attributed to Du Bois Reymond, is proved in a very simple manner, although perhaps a little too cleverly for the unsophiscated reader. The Hilbert independent integral, the concept of a field and related ideas and theorems are introduced in each of chapters II, III, IV, in the special forms which they take for the problems there treated. The usual methods of the sufficiency proofs are exemplified in a similar manner, as are also the cases in which one or two endpoints are variable on arbitrary curves. In the discussion of these latter questions the author is not satisfied to treat merely first order conditions, but gives a complete account of the theory of the focal point, which receives in each of the special cases a more or less simple geometrical interpretation. The proof of the existence of a unique extremal through two given points and the construction of a field in the brachistochrone problem are carried out by drawing in a very skillful way upon well known properties of the cycloid. In connection with the same problem we are introduced to the envelope theorem and to the geometric proof of Jacobi's theorem which depends upon it.

Particularly valuable is the presentation in the fourth chapter of the results obtained by Sinclair and MacNeish in the catenary problem, not heretofore available in a connected form, and the clear discussion of the relation between the catenary and the Goldschmidt straight line solutions of the problem of the surface of revolution of minimum area.

The final chapter, more than any of the others, is of the usual textbook character. The reader who has followed the discussions of the three special problems in the preceding chapters should be well prepared to understand now the general theory as here presented. He meets again, in the form of general theorems, statements with whose general character he has had an opportunity to become acquainted. But the chapter contains more than these generalizations of the results previously obtained for special cases. It is here that we learn for the first time of the distinction between weak and strong minima, of the Weierstrass E-function, of Legendre's condition, of the Weierstrass-

Erdmann corner point condition, etc., which for various reasons have not been brought forward before. It is furthermore made clear that the geometrical treatment of the conjugate point condition is not sufficient in all cases. This leads to a discussion of the second variation by the elegant method which the author introduced some years ago and which has since been applied by several of his pupils to a variety of more general problems, but a systematic exposition of which has not hitherto been available outside the journals. Moreover, the reader who merely wants to get the results will find an accurate statement of sets of necessary and of sufficient conditions for the cases both of fixed and of variable endpoints. The chapter closes in the same key in which the first one opened, viz., with historical remarks.

Enough has been said to make clear that in the judgment of the reviewer, Professor Bliss has carried through a difficult task with remarkable success. We have here a new type of mathematical book. It is not a textbook, neither is it addressed to the specialist, actual or *in spe.* It is intended for that very indefinite group, the intelligent general public. This group surely includes teachers of mathematics, even those who may have taken a course in the calculus of variations! Certainly engineers who expect to do more than routine work, physicists and chemists and other workers in natural science can profit largely from familiarizing themselves with the leading ideas set forth in this book. To many it should be of immediate usefulness.

But to write on an advanced mathematical topic for a public of such mixed and uncertain qualifications is much like lecturing to a radio audience: one never can tell whether there is any real catching on. In his selection of material the author had to trust to his judgment; the style of treatment is in a large measure determined by this choice. Now one can readily conceive of other selections which might have been made. A treatment of an isoperimetric problem, mention at least of some problems leading to integrals with more unknown functions, and to extrema of multiple integrals would seem to me to be necessary in order to give an adequate picture of the broad scope of the calculus of variations. This remark however amounts to little else than to asking for "more of the same kind"; for I should not want any of these topics to replace anything that is in the book now. And this indeed is my chief criticism; the book is too short. It is a fine beginning; but it should be continued.

This then is the remark with which I wish to close this review: Let the series of Carus Monographs, for which I anticipate a long and successful career, develop a semi-periodic character which will make it possible to carry forward to other parts of the field the discussion of the calculus of variations which Professor Bliss's volume has so excellently initiated.

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