SOME PROBLEMS OF CLOSURE CONNECTED WITH THE GEISER TRANSFORMATION*

BY ARNOLD EMCH

1. *Introduction.* Problems of closure may be defined as series of geometrical operations of the same type performed on a given figure with the property that the series closes after a finite number of steps and that the closure in one instance has as a consequence the closure of an infinite number of series with the same number of steps performed on the same given figure. As examples of such problems may be mentioned the well known Steiner series of circles attached to two given non-intersecting circles, the Poncelet polygons, the Steiner polygons inscribed in a cubic, etc. There are various methods of treating problems of this kind. One very effective method for a certain class of problems is by means of elliptic functions, as inaugurated by Jacobi† and Clebsch.‡

Another method, distinguished by its simplicity and directness, has been established by A. Hurwitz§ and is based on the correspondence principle in a one-parameter algebraic domain. For example, if the correspondence between the elements is (m, n) on a rational curve, there are $m+n$ coincidences. Now it is possible that in certain cases the correspondence may be such that there are more than $m+n$ coincidences. If this happens, then there are an infinite number of such coincidences and we have a

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† *Über die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie,* CRELLE'S JOURNAL, vol. 3, p. 376.

‡ *Über einen Satz von Steiner und einige Punkte der Theorie der Curven dritter Ordnung,* CRELLE'S JOURNAL, vol. 63 (1864), pp. 94–121.

§ *Über unendlich-vieldeutige geometrische Aufgaben, insbesondere über die Schliessungsprobleme,* MATHEMATISCHE ANNALEN, vol. 15 (1879), pp. 8–15.

problem of closure. Algebraically the problem may be stated thus: If the solution of a geometrical problem leads to an equation of degree n in one unknown (parameter), and if under certain imposed conditions this equation admits of more than n roots, then the equation has an infinite number of roots, and the problem admits of an infinite number of solutions.

Recently Professor A. B. Coble has worked out a method of procedure of an invariantive character by which he has been able to solve problems of closure (porisms) in a very elegant manner.* The purpose of this paper is to establish some new problems of closure by a certain mapping process applied to previously known problems of closure. It was also by a mapping process that the writer found the problems of closure stated in the recently published paper on the geometry of the symmetric group.†

2. *The Geiser Transformation.*‡ For a better understanding of what follows it is perhaps well to state the principal properties of this well known involutory Cremona transformation in a plane in agreement with our notation. The seven base-points $A_1, A_2, \cdots, A_7$ in a general position determine a net of cubics so that any two cubics of the net intersect in two points P, P' which as a pair of corresponding points define the Geiser transformation. The base-curves C_i are nodal cubics through the base-points, with their nodes at the A_i's respectively. The Φ_i's are octics with triple points at each of the A_i's, and determine a net of octics. The transformation has the pointwise invariant sextic C_6^i with double points at each of the A_i's,

* *Multiple binary forms with the closure property,* American Journal, vol. 43 (1921), pp. 1–19.

† American Journal, vol. 45 (1923), pp. 192–207. See also the author's monograph, *Applications of elliptic functions to problems of closure,* The University of Colorado Studies, vol. 1 (1902), pp. 81–133.

‡ Geiser, *Über zwei geometrische Probleme,* Crelle's Journal, vol. 67 (1867), pp. 78–89; Sturm, *Die Lehre von den Geometrischen Verwandtschaften,* vol. 4, 1909, pp. 96–103.

which is consequently of genus 3, and which is the Jacobian of the net of cubics. It has the same nodal tangents at the A_i's as the nodal base-cubics. The transformation is of class 1. The invariant isologue curve C_S attached to a point S is an elliptic cubic which outside of the A_i's cuts the C_6^i in four points P_1, P_2, P_3, P_4 which form a Steinerian quadruple on the cubic C_S; i. e., the tangents to the C_S at the P_i's meet in S which is on the C_S. From this follows that corresponding pairs P, P' of the Geiser transformation on the C_S are also corresponding pairs in an involutory quadratic transformation with P_1, P_2, P_3, P_4 as the quadrangle of invariant points. This property of the C_S makes it easy to establish the relation between an invariant C_μ of the Geiser transformation and the corresponding class-curve K_ν from which it is generated.

The order of the transformed C_μ is in general 8μ. Hence in order that this number reduce to μ it is necessary that the C_μ have multiplicities j_i at the A_i's, such that $\sum 3ij_i = 8\mu - \mu = 7\mu$, which shows that μ must be a multiple of three. The class ν of the K_ν is equal to the number of lines joining couples of corresponding points P, P' on the C_μ through any given points S. All such couples also lie on the attached C_s. The intersections of the C_μ and the C_s at the A_i's absorb $\sum j_i$ points, so that outside of the A_i's there are $3\mu - \sum j_i$ intersections which arrange themselves into half that many couples aligned through S. Hence $\nu = (1/2)(3\mu - \sum j_i)$. In case of an invariant C_3, $\nu = 1$, which corroborates the fact that the net of cubics through the A_i's is identical with the net of isologue cubics.

Let us consider an invariant C_{3m} in the Geiser transformation. As for a general C_{3m} the corresponding curve is of order $24m$, a curve of order $21m$ must split off in order to have a proper corresponding curve of order $3m$. Assuming the simple case in which the C_{3m} has equal multiplicities at the A_i's, these multiplicities are necessarily of order m, since whenever a branch of the C_{3m} passes through an A_i, the corresponding base-cubic splits off as a part of the C'_{3m}. Now the isologue of a point S cuts the C_{3m} in $3 \cdot 3m - 7 \cdot m = 2m$ points outside of the A_i's. Hence for an invariant C_{3m} these arrange themselves into m couples of corresponding points on lines through S. Hence the invariant C_{3m} may be generated from a K_m.

As the K_m is determined by $m(m+3)/2$ conditions, the manifold of invariant C_{3m}'s of this type is equal to this number. The number of conditions of the class of C_{3m}'s with multiplicities of order m at each A_i is $k = (1/2)\,3m(3m+3) - 7(1/2)m(m+1) = m(m+1)$. The question is, how must the k points be chosen to insure an invariant C_{3m}. It is obvious that since k is even, a sufficient condition is that the k points form $k/2$ couples. But this is not always necessary. From the k points choose γ couples of corresponding points, then the C_{3m} and C'_{3m} have at least $7m^2 + 2\gamma + 4m$ points in common, since the C_{3m} cuts the pointwise invariant sextic in $6 \cdot 3m - 7 \cdot 2 \cdot m = 4m$ points. If $7m^2 + 2\gamma + 4m \geqq 9m^2$, then $C_{3m} \equiv C'_{3m}$. This condition reduces to $2\gamma \geqq 2m^2 - 4m$. On the other hand $2\gamma \leqq m(m+1)$. For $m = 5$, this gives $30 \leqq 2\gamma \leqq 30$. Thus in case of an invariant C_{15}, 15 couples determine such a curve uniquely. As $2m^2 - 4m - (m^2 + m) = m(m-5)$ is positive for all values of $m > 5$, in case of C_{3m}'s for $m > 5$, $(1/2)m(m+1)$ couples determine the invariant C_{3m} uniquely.

As an example of particular interest may be mentioned the class of invariant sextics with double points at each of the base-points. In the first place a sextic with double points at each of the A_i's depends on six free constants and is transformed into a sextic of the same type since $6 \cdot 8 - 2 \cdot 7 \cdot 3 = 6$. Such a non-invariant sextic C_6^* cuts the C_6^i in $36 - 7 \cdot 4 = 8$ points J outside of the A_i's. The C_6^* and the C_6^i determine a pencil of sextics with the same multiplicities at the A_i's and all passing through the 8 points J. Through every point of C_6^i there is a selfcorresponding direction, i. e., a line on which lie two corresponding points of the Geiser transformation infinitely close to the C_6^i. Consider now any of the 8 points J and the corresponding line-element through J. There is just one sextic C_6^* of the pencil through the A_i's and the J's which will have this element as a tangent at J. The transformed $C_6^{*\prime}$ belongs to the pencil, since the J's are invariant, and along the element through J cuts the C_6^* in two consecutive points. Hence the two curves intersect in 37 points and are therefore identical. This establishes the existence of invariant sextics. As every invariant sextic C_6 determines a pencil of non-invariant curves and as all non-invariant curves are contained among such pencils, the system of invariant sextics with double points at the A_i's depends on five effective constants. This may be verified as follows: An invariant cubic C_3 cuts an invariant C_6 in four points outside of the A_i's. These lie in couples of corresponding points on two lines through S. Consequently the lines joining corresponding points on an invariant C_6 with double points at the A_i's envelope a conic K_2. Conversely any invariant sextic of this sort may be generated from a conic K_2 of class two. There are therefore ∞^5 such sextics. If we denote by ψ_1, ψ_2, ψ_3 three linearly independent invariant cubics, any invariant sextic of the system may be represented in the form $\sum_{i,k=1}^{3} a_{ik}\psi_i\psi_k = 0$. To sum up we have the following theorem.

THEOREM I. *The entire class of ∞^5 invariant sextics with double points at the base-points may be generated from the class of conics K_2. An invariant sextic C_6 cuts the pointwise invariant sextic C_6^i in 8 points, so that the tangents to the C_6 at these points touch a conic (K_2).*

The pointwise invariant curve C_6^i connected with seven generic points in a plane is sometimes called the *Aronhold* curve. It cannot be represented as a polynomial of the second degree of three linearly independent cubics through the seven points. However its square may be expressed as a certain polynomial of the fourth degree in the three cubics $C_3^{(1)}$, $C_3^{(2)}$, $C_3^{(3)}$, so that $(C_6^i)^2 = F(C_3^{(1)}, C_3^{(2)}, C_3^{(3)})$, where F is a general quartic with the C_3's as projective coordinates. Hence the general quartic can be resolved by three linearly independent cubics through seven points.

It is easy to study higher systems of invariant curves in the Geiser transformation by the properties of the transformation and by the curves K_ν from which they are generated. A K_ν generates a reducible curve of order 9ν with an equation of the form $(C_6^i)^\nu H(\psi_1, \psi_2, \psi_3) = 0$, in which the H is of degree 3ν in the x's. The pointwise invariant curve C_6^i splits off ν times and leaves an invariant curve H of order 3ν with ν-fold points at the base-points.

3. *Mapping by the Geiser Transformation on a General Cubic Surface.* Consider the cubic Cremona transformation between two quaternary spaces $\sum$ and $\sum'$ in which to the planes p of $\sum$ correspond cubic surfaces C' through the base-curve S' in $\sum'$ and conversely to the planes p' of $\sum'$ correspond cubic surfaces C through the base-curve S in $\sum$. The curves S and S' are sextics of genus three and the base-surfaces of the transformation are octic surfaces formed by the trisecants of S and S'. To a point of S corresponds a trisecant of S', and conversely to a point of S' a trisecant of S.

Choose a definite plane p in $\sum$ as the plane of a Geiser transformation which by the cubic transformation is mapped into a definite cubic C' of $\sum'$. The sextic S cuts p in six points $A_1, \ldots, A_6$ which in general do not lie on a conic. These points we take as six points of the Geiser transformation, while A_7 may be chosen in some fixed position independent of the six other points. To $A_1, \ldots, A_6$ correspond on C' six lines $a_1', \ldots, a_6'$, while the image of A_7 is some point A_7' on C'. To a plane section p' through A_7' with C' corresponds in p a plane cubic C_3^p through $A_1, \ldots, A_7$. Likewise to another plane section q' through A_7' corresponds in p another plane cubic C_3^q through the base-points. C_3^p and C_3^q intersect, outside of the A's, in two points P and Q which are corresponding points in the Geiser transformation. To them correspond on C' two points P' and Q' which lie in p' and q', and which are therefore the points in which the line of intersection of p' and q' cuts the cubic surface C'. Thus to pairs of corresponding points of the Geiser transformation correspond on the cubic C' pairs of corresponding points cut out by secants u' through A_7'. Every line in p carries one pair of corresponding points, and to it corresponds in $\sum'$ uniquely a line u' through A_7'.

THEOREM II. *The correspondence between u and u' is* (1,1) *and involutory. Hence the points P and lines u of p are in involutory reciprocity with the planes p' and lines u' through A_7'. In this reciprocity, to a curve K_m of class m in p corresponds in $\sum'$ a cone K_m' of order m with A_7' as a vertex. The locus of pairs of corresponding points on tangents u' of K_m is an invariant C_{3m} of the Geiser transformation, as has been proved in* § 2. *The cone K_m' cuts C' in a curve C_{3m}' of order* $3m$ *which is invariant in the involution (P', Q') on C'. Thus C_{3m} and C_{3m}' correspond to each other in the cubic transformation between $\sum$ and $\sum'$.*

To the plane sections through A_7' and a_i' correspond in p the base-curves of the Geiser transformation. To the plane section of the tangent-plane to C' at A_7' corresponds the base-curve with A_7 as a node. To the tact-sextic of the tangent cone from A_7' to C' corresponds in p the Aronhold curve or the pointwise invariant sextic or Jacobian of the net of plane cubics through the A's.

4. *Problems of Closure.* A cone K_m' with A_7' as a vertex cuts the cubic C' in a curve C_{3m}' of order $3m$. As K_m' cuts each of the lines a_i' in m points, the corresponding C_{3m} in p, generated from the class-curve K_m, has $A_1, \ldots, A_6$ as multiple points of order m. But the C_{3m}' has an m-fold point at the point A_7', so that also A_7 has the multiplicity m on C_{3m}. A tangent-plane to the cone K_m' along an element u' cuts C' in a cubic C_3' which touches C_{3m}' in the two points P' and Q' cut out by u'. Thus, to C_3' corresponds in p a cubic C_3 which touches C_{3m} in corresponding points of the Geiser transformation.

As a particular case, choose two quadric cones K_2' and L_2' with common vertex at A_7', but not intersecting in real generators (u'). They intersect C' in two sextics T' and S'. Every tangent plane of K_2' cuts C' in a cubic C_3' which touches T' in two points on a u' and cuts S' in two pairs of points whose joins also pass through A_7', since they lie on generators of L_2'. If we now construct a pyramid with vertex at A_7' such that its faces are tangent to K_2' and its edges are generators of L_2', and if once such a pyramid closes for two fixed cones K_2' and L_2', then there are an infinite number of such pyramids, circumscribed to K_2' and inscribed to L_2'. If by means of the cubic transformation we go back to the Geiser plane p, we derive immediately the following theorem.

THEOREM III. *Let T and S be two invariant sextics of the Geiser transformation and C_1 an invariant cubic touching T in two points T_1 and T_1' and cutting S in two couples of corresponding points S_1, S_1' and S_2, S_2'. Through S_2, S_2' pass an invariant cubic C_2 touching T in a couple of corresponding points T_2, T_2' and cutting S in a couple S_3, S_3'. Through S_3, S_3' pass a cubic C_3 touching T in T_3, T_3' and cutting S in S_4, S_4', and so forth. Suppose this process continued n times, so that the last cubic C_n of the series cuts S in S_{n+1}, S_{n+1}'. If once S_{n+1}, S_{n+1}' coincides with S_1, S_1, i. e., if the process closes, then there is always closure after n operations, no matter what initial cubic C_1 touching the sextic T in a couple T_1, T_1' is chosen.*

The joins of all couples T_i, T_i' envelope a conic K_2, those of S_i, S_i' a conic L_2, such that K_2 and L_2 are correlative to K_2' and L_2'.

A general cubic cone K' with vertex at A_7' cuts the cubic surface C' in a curve S_9' of order 9. To it corresponds in p an invariant curve of the same order, generated from a correlative plane cubic of class 3. Through a generator u' of K' pass two tangent planes α_g and α_h touching K' along the generators g and h. α_g and α_h cut

C' in plane cubics C'_g and C'_h which pass through the intersection U and U' of u' with C', and which touch the C'_9 in couples G, G' and H, H' of corresponding points in the involution on the cubic C'. The closure property of the quadrilateral pyramid with vertex at A'_7 whose six edges are generators of K', all in analogy with the closure theory of Steiner polygons inscribed in a plane cubic, transferred to the Geiser plane leads to the following theorem.

THEOREM IV. *Given an invariant S_9 in the Geiser transformation and on it a couple (U, U') of corresponding points. Through (U, U') draw two invariant cubics touching S_9 in two couples (G, G') and (H, H') respectively. Through (G, G') pass any invariant cubic C_1 cutting S_9 in two couples (A, A') and (B, B'); through (H, H') and (B, B') pass an invariant cubic C_2 cutting S_9 in (C, C'); through (G, G') and (C, C') pass the invariant cubic C_3 cutting S_9 in (D, D'); through (H, H') and (D, D') pass the invariant cubic C_4; then C_4 will always pass through (A, A'), no matter what initial first couple (A, A') or initial cubic C_1 is chosen.*

By the mapping process explained above a number of problems of closure may be obtained without difficulty. Thus we may state the following theorem.

THEOREM V. *Given an invariant curve C_9 of order* 9 *with triple points at the base-points of the Geiser transformation. Through a couple of corresponding points of C_9 draw the four invariant cubics touching the C_9, each touching the C_9 in a couple of corresponding points. Thus four couples are obtained which we define as a Steinerian octuple on the C_9. Consider a second octuple of this sort on the C_9. Any couple of the first octuple together with any couple of the second octuple determine an invariant cubic uniquely. In this manner are determined* 16 *invariant cubics which by fours intersect in four couples of a third octuple on the C_9.*

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