# NUCLEAR AND HYPER-NUCLEAR POINTS IN THE THEORY OF ABSTRACT SETS*

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1. *Introduction.* In his note, *Le théorème de* Borel *dans la theorie des ensembles abstraits,*† Fréchet considers the problem: determine the most general class (L) for which the theorem of Borel holds true. This class is found to be a class (S), that is, a class (L) in which the derived set of every set is closed. At the end of the note he calls attention to the fact that the stronger theorem of Borel-Lebesgue may not hold in a given class (S) and proposes the question: what is the most general class (L) for which we may state the theorem of Borel-Lebesgue? That such a class (L) be a class (S) is necessary but not sufficient.

This attracted the attention of R. L. Moore,‡ who showed by the aid of the theory of transfinite ordinals that the most general class (L) which admits the theorem of Borel-Lebesgue is a class (S) with the further property "every compact set is perfectly compact". The property *perfectly compact,* so named by Fréchet,§ is defined as follows. A set $E$ is perfectly compact if every monotone sequence of subsets of $E$ determines an element which is common to all the sets of the sequence or to their derived sets. A sequence of sets is monotone if of any two sets of the sequence one contains the other.

Later Fréchet,|| developing the theory of classes (V)

* Presented to the Society, December 1, 1923.

† Bulletin de la Société de France, vol. 45 (1917), pp. 1–8. Called Fréchet, I hereafter.

‡ *On the most general class* (L) *of* Fréchet *in which the Heine-Borel-Lebesgue theorem holds true*, Proceedings of the National Academy of Sciences, vol. 5 (1919), pp. 196–210.

§ *Sur les ensembles abstraits,* Annales de l'Ecole Normale (3), vol. 38 (1921), p. 342. Called Fréchet, II hereafter.

|| *Sur la notion de voisinage dans les ensembles abstraits,* Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 138–156. Called Fréchet, III hereafter.

more general than classes (L), resumed the study of the theorem of Borel. This theorem was found for classes (V) to be the equivalent of the property of Hedrick; that is, for every set $E$ in the class (V), element $p$ interior to $E$, and set $F$ which has $p$ for a limit element, there is a subset $F_1$ of $F$ such that every element of $F_1$ is interior to $E$.* This property implies the closure of derived classes. The theorem of R. L. Moore was extended by Fréchet to these more general classes (V) with the following results. The property of Borel-Lebesgue implies the property perfectly compact in the most general class (V). Under the hypothesis of the property of Hedrick, every set which is perfectly compact in itself possesses the property of Borel-Lebesgue.†

Kuratowski and Sierpinski‡ presented another solution of the problem. An element or point $p$ of a set $E$ is of power $\mu$ relative to $E$ if there is a subset of $E$ of power $\mu$ *interior* to every neighborhood of $p$, but not every neighborhood of $p$ contains a subset of $E$ of power greater than $\mu$. It is convenient to say that a point $p$ of a set $E$ of power $\mu$ is a *hyper-nuclear* point of $E$ if $p$ is of power $\mu$ relative to $E$. A necessary and sufficient condition that the theorem of Borel-Lebesgue hold in a class (L) is that every infinite compact set whose derived set is also compact determine at least one hyper-nuclear point. Since in a class (L) the properties perfectly compact, and derived sets are closed, together form a necessary and sufficient condition for the theorem of Borel-Lebesgue it follows that in a class (L) they are equivalent to the condition of Kuratowski and Sierpinski. Section 4 of the present paper contains a generalization of the theorem of Borel-Lebesgue which holds in the most general class (V) and reduces to the theorem of Kuratowski and Sierpinski when

* Fréchet, III, p. 155.

† Fréchet, II, pp. 346–49, §§ 8–10.

‡ *Le théorème de Borel-Lebesgue dans la théorie des ensembles abstraits,* Fundamenta Mathematicae, vol. 2 (1921), pp. 172–78.

the class (V) is equivalent to a class (L). This new theorem is shown by the example of § 5 to be independent of the closure of derived sets.

Fréchet* has shown that in a class (D), that is a class admitting a generalization of distance, every compact set is perfectly compact, and proposes the problem: determine the most general class (V) with this property. Section 4 below presents in terms of the concept *nuclear* point† a necessary and sufficient condition that every compact subset of a class (V) be perfectly compact.

2. *On Terminology.* As a basis for the following discussion we postulate a space $P$ of points $p$ and a system of families of subsets $V$ of $P$ called neighborhoods. To each point $p$ is assigned in a definite way a family of neighborhoods $V_p$. There is no loss of generality in assuming that $V_p$ contains $p$, and for convenience we shall make that assumption. A point $p$ is a limit point of a set $E$ if every neighborhood of $p$ contains a point of $E$ distinct from $p$. The relation limit point so defined has the following properties.

1) Every limit point of a set $E$ is a limit point of every set containing $E$.

2) Whether $p$ is a limit point of a set $E$ or not depends only on the elements of $E$ other than $p$‡.

A point $p$ is interior to a set $G$ if $p$ is an element of $G$ and if $G$ contains a point of every set which has $p$ for a limit point. Then $p$ is interior to every neighborhood $V_p$. Furthermore if $p$ is interior to $G$ then some neighborhood of $p$, $V_p$, is a subset of $G$.

A series $S$ of sets $G$ is called monotone if of any two sets $G$, $G'$ of the series one is a subset of the other. A monotone series will be called *closed* if the sets $G$ or their derived sets have a common element; otherwise *open*. If

* Fréchet, II, p. 346.

† A point $p$ is a nuclear point of a set $E$ of power $\mu$ if every neighborhood of $p$ contains a subset of $E$ of power $\mu$.

‡ Fréchet, III, p. 140.

each such common element belongs to a set $E$ the series will be said to be closed in $E$.

A set $E$ is perfectly compact if every monotone series $S$ of subsets $G$ of $E$ is closed; perfectly self-compact, if every such series $S$ is closed in $E$. Fréchet has shown that every perfectly self-compact set is compact and that every perfectly self-compact set is self-compact*.

3. *Procedure under the Zermelo Axiom.* The following procedure based upon the assumption of the Zermelo axiom and the well known properties of the transfinite ordinal numbers was employed by Kuratowski and Sierpinski† and is the basis for several of the following proofs. Let $Q$ be an aggregate of power $\mu$ and of elements $q$. Of the transfinite ordinals $\Omega$ for which the aggregate of all ordinals $\alpha < \Omega$ has the power $\mu$ there is a least, $\Omega_0$. Let

$$(1) \quad q_1, q_2, q_3, \ldots, q_n, \ldots, q_\omega, \ldots, q_\alpha, \ldots \; (\alpha < \Omega_0)$$

represent a 1—1 correspondence between the aggregate $Q$ and the aggregate of all ordinal numbers $\alpha < \Omega_0$. If a sequence of ordinals $\beta < \Omega_0$ is determined so that for every $\alpha$ there is a $\beta > \alpha$ then the $\beta$'s form a series of the ordinal type $\Omega_0$ and the aggregate of all such ordinals $\beta$ is of power $\mu$. Several of the proofs in the sequel depend upon a correspondence of the type (1) and the further fact that the elements $q_\beta$ form an aggregate of power $\mu$.

4. *Nuclear Points.* A point $p$ will be said to be a *nuclear point* of a set $E$ in case every neighborhood $V_p$ contains a subset $H$ of $E$ equivalent to $E$; that is, of the same power or cardinal number as $E$.

THEOREM 1. *If an infinite set $E$ is perfectly compact $E$ determines at least one nuclear point.*

Let $E$ be an infinite set of points $p$ and denote the cardinal

* Fréchet, II, p. 343. A set $E$ is compact if every infinite subset of $E$ has a limit point; self-compact if it has a limit point in $E$.

† Loc. cit., p. 175.

number of $E$ by $\mu$. Then as in § 3, we have an ordinal $\Omega_0$ and a 1—1 correspondence

$$(2) \qquad p_1, p_2, p_3, \ldots, p_n, \ldots, p_\omega, \ldots, p_\alpha, \ldots \quad (\alpha < \Omega_0)$$

between the ordinal numbers $\alpha$ and elements $p$ of $E$. Let

$$G_\alpha = \sum_{\beta \geqq \alpha}^{\Omega_0} p_\beta.$$

Then $G_1 = E$. The $G_\alpha$ form a monotonic sequence of subsets of $E$ with no common element. But $E$ is perfectly compact, so there must exist an element $q$ common to all the $G'_\alpha$.

Let $V$ be a neighborhood of the point $q$. If $q$ is not in $E$ then it is a limit point of every $G_\alpha$ and therefore $V$ contains a point $q_\alpha$ of $G_\alpha$ distinct from $q$. If $q$ is in $E$ there is an $\alpha$ such that $G_\alpha$ does not contain $q$. For every $\alpha' \geqq \alpha$ the point $q$ is a limit point of $G_{\alpha'}$. Again $q$ is a limit point of every $G_\alpha$, and consequently $V$ contains a point $q_\alpha$ of $G_\alpha$. Let $Q$ be the set of all distinct $q_\alpha$, and for a given element $q_\alpha$ of $Q$ let $\beta$ denote the index such that $q_\alpha$ is in $G_\beta$ but not in $G_{\beta+1}$. The index $\beta$ is in fact that index $\alpha$ which corresponds to $q_\alpha$ regarded as an element of $E$ and determined by the correspondence (2). These indices $\beta$ are such that for every $\alpha < \Omega_0$ there is a $\beta > \alpha$. It follows at once that the $\beta$'s and therefore the points of $Q$ form an aggregate of power $\mu$. Therefore $q$ is a nuclear point of $E$.

COROLLARY. *Every set $E$ which is perfectly self-compact contains a nuclear point.*

For the point $q$ of the preceding proof may be assumed to be a point of the set $E$.

THEOREM 2. *If every infinite subset of a set $E$ of points of the space $P$ determines a nuclear point then $E$ is perfectly compact.*

Let $S$ be an open monotonic sequence of subsets $G$ of a set $E$, satisfying the hypothesis of Theorem 2. Let $H$ be a set of points $p$ such that every $G$ contains a point of $H$.

Let $H$ be of power $\mu$ and let $\Omega_0$ be the least ordinal of that power. We may assume a $1-1$ correspondence of the type (2) between the elements of $H$ and the ordinals $\alpha < \Omega_0$. For each element $p_\alpha$ we may select a set $G_\alpha$ which does not contain $p_\alpha$ and such that $G_{\alpha+1}$ is contained in $G_\alpha$. Since every set $G$ of the series $S$ contains some element of $H$ it follows that every $G$ contains some $G_\alpha$. Now by hypothesis $H$ has a nuclear point $q$. Let $V$ be a neighborhood of $q$ and let $Q$ denote the subset of $H$ of power $\mu$ in $V$. Then if $\beta$ is any ordinal, $\beta < \Omega_0$, there is an element $p_\alpha$ of $H$ in $Q$ for which $\beta < \alpha$. Otherwise, because of the hypothesis on $\Omega_0$, $Q$ would be of power less than $\mu$. It follows at once that $V$ contains an element of every $G_\alpha$ and therefore of every $G$. Consequently $q$ is common to all the $G'$, contrary to the hypothesis that $S$ is an open sequence.

Theorems 1 and 2 imply the following theorem.

THEOREM 3. *A necessary and sufficient condition that a set $E$ be perfectly compact is that every infinite subset of $E$ determine at least one nuclear point.*

COROLLARY. *A necessary and sufficient condition that a class $(V)$ be a class $(M)$; that is, that every compact set $E$ be perfectly compact, is that every infinite compact set $E$ possess at least one nuclear point.*

For if a compact set is finite it is perfectly compact. If it is infinite the result follows from Theorem 3.

5. *Hyper-Nuclear Points.* The concept *hyper-nuclear point* is helpful in generalizing the theorem of Kuratowski-Sierpinski. A point $p$ is a *hyper-nuclear point* of a set $E$ of power $\mu$ in case there is a subset $H$ of $E$ of power $\mu$ interior to every neighborhood of $p$.

The theorem to be generalized may be stated as follows. A necessary and sufficient condition that the theorem of Borel-Lebesgue apply to a class (L) is that every infinite compact set $E$ whose derived set $E'$ is also compact possess at least one hyper-nuclear point.

The extension of this result to classes (V) in general is a consequence of the two following theorems.

THEOREM 4. *If a self-compact set $H$ contains a hyper-nuclear point of every infinite subset of $H$, then $H$ admits the property of Borel-Lebesgue.**

Let us suppose that $H$ does not admit the property of Borel-Lebesgue. Then there is a family $F$ of sets $I$ which covers $H$ and contains no finite subfamily with the same property. Among such families $F$ there is (according to the theorem of Zermelo) at least one, say $F_0$, whose power $\mu$ is a minimum. Let $\Omega_0$ be the first transfinite ordinal of power $\mu$. There is a well ordered set

(3) $\quad I_1, I_2, I_3, \ldots, I_n, \ldots, I_\omega, \ldots, I_\alpha, \ldots \; (\alpha < \Omega_0)$

of order type $\Omega_0$ comprising the totality of the sets $I$ of $F_0$ and in one-one correspondence with the numbers $\alpha < \Omega_0$.

From the definition of $F_0$, every point of $H$ is interior to at least one of the sets $I_\alpha$ but no segment of the sequence (3) has this property. We can also suppose that every set $I_\alpha$ contains in its interior an element of $H$, denoted by $p_\alpha$, which is not interior to any set $I_\xi$ for $\xi < \alpha$, since all the sets $I_\alpha$ which do not have this property could be suppressed in the sequence (3) without reducing the ordinal type (because there is no family covering $H$ of power less than $\mu$, and the reduced family would not fail to cover $H$).

The set $Q$ of all the points $p_\alpha$ is evidently of the power $\mu$ (since $p_\alpha \neq p_\beta$ for $\alpha < \beta$; $p_\alpha$ being interior to $I_\alpha$ while $p_\beta$ is not). But $Q$ is a subset of $H$ and so by hypothesis $H$ contains a hyper-nuclear point $q$ of $Q$. This point $q$ is interior to some one of the sets $I_\alpha$ of the sequence (3), let it be $I_\gamma$. Since $q$ is a hyper-nuclear point of $Q$ there is a subset of $Q$ of power $\mu$ interior to every neighborhood of $q$. But $q$ is interior to $I_\gamma$ which must contain a neighborhood $V$ of $q$. It follows at once that there is a subset of

* The proofs of Theorems 4 and 5 differ from the corresponding proofs of Kuratowski and Sierpinski (loc. cit., pp. 174–75) only with respect to those details which are involved in the generalization.

$Q$ of power $\mu$ interior to $I_\gamma$. But the totality of the points $p_\xi$ for $\xi \leqq \gamma$ is of cardinal number less than $\mu$. Therefore $I_\gamma$ contains in its interior an element $p_\nu$ of $Q$ whose index $\nu$ exceeds $\gamma$. But by definition $p_\nu$ is not interior to any set $I_\xi \, (\xi < \gamma)$. This is a contradiction, and the assumption that $H$ does not possess the property of Borel-Lebesgue fails.

THEOREM 5. *If an aggregate $H$ admits the property of Borel-Lebesgue, $H$ is self-compact and contains a hyper-nuclear point of every infinite subset of $H$.*

That the aggregate $H$ of the theorem is self-compact is well known.* Let $Q$ be an infinite subset of $H$ of power $\mu$ and suppose that to each point $p$ of $H$ there is a neighborhood $V_p$ such that the subset of $Q$ in $V_p$ is of power less than $\mu$. The totality of such neighborhoods $V_p$ covers $H$, and, by the Borel-Lebesgue property, it can be replaced by a finite family with the same property. Then $Q$ can be represented as the sum of a finite number of sets each of power less than $\mu$, contrary to the hypothesis that $Q$ is of power $\mu$.

From Theorems 4 and 5 we have the following theorem.

THEOREM 6. *A necessary and sufficient condition that an aggregate $H$ possess the property of Borel-Lebesgue is that $H$ be self-compact and that every infinite subset of $H$ be hyper-nuclear in $H$.*

It should be noted that the hypothesis "derived sets are closed" does not enter this theorem directly. That for classes (V) in general the theorem above is independent of the closure of derived classes is shown by the example of the following section.

In a class (L) every self-compact class is compact and closed. From the theorem of Kuratowski and Sierpinski and Theorem 6 above we have the following theorem.

THEOREM 7. *In a class* (L) *a necessary and sufficient condition that every infinite compact set $E$ whose derived set is compact determine a hyper-nuclear point is that every closed compact set $H$ have the property "every infinite subset of $H$ has a hyper-nuclear point in $H$".*

* Fréchet, III, p. 152, § 19.

Since Fréchet has shown that every set with the property of Borel-Lebesgue is perfectly self-compact it follows from Theorem 6 that we have the following theorem.

THEOREM 8. *If a set $H$ is self-compact and contains a hyper-nuclear point of every infinite subset of $H$, then $H$ is perfectly self-compact.*

This theorem should be compared with Theorem 2. It would be interesting to have an example showing that the conclusion does not hold if the word "hyper-nuclear" in Theorem 8 is replaced by the word "nuclear".

6. *An Example.* It has been shown that in a class (L) the theorem of Borel implies the closure of derived classes but that in the more general classes (V) the theorem of Borel and that of Borel-Lebesgue may be formulated without the use of the hypothesis of the closure of derived sets. The following example completes the proof of this independence.

Let $P$ be the class of all number pairs $p=(n,m)$ where $n=1,2,3,\cdots$; $m=1,2,3$, together with the number pair $(0,0)$. The neighborhoods $V$ are defined as follows. The elements $p=(n,1)$, $n=1,2,3,\cdots$, have but one neighborhood consisting in each case of the element $p$ alone. Each element of the form $p=(n,2)$ has a family of neighborhoods $V_{kp}$, consisting for given $k$ of the point $p$ and all points $q=(n,1)$ for which $n\geqq k$ $(k=1,2,3,\cdots)$. The elements $p=(n,3)$ have a family of neighborhoods $V_{kp}$, each comprising the point $p$ and all points $(n,1)$ and $(n,2)$ for which $n\geqq k$ $(k=1,2,3,\cdots)$. The neighborhoods $V_{kp}$ of the point $p=(0,0)$ contain all the points $(n,2)$ and $(n,3)$ for $n=k$ $(k=1,2,3,\cdots)$, in addition to the point $p=(0,0)$ itself. It is easy to see that $P$ is compact, and that every infinite subset of $P$ determines a hyper-nuclear point. It may be verified directly that although derived sets are not closed, the theorem of Borel holds. The theorem of Borel-Lebesgue is satisfied vacuously, since the class $P$ is enumerable.

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