ON SIMPLE GROUPS OF LOW ORDER*

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1. *Introduction.* The known simple groups of composite order were tabulated by Dickson.† All the groups there enumerated as far as order 7920 belong to well known infinite systems. The exhaustive determination of the orders of simple groups was carried by Siceloff‡ and his predecessors as far as order 3640. Recently G. A. Miller§ has shown that there is only one type of simple group of order 2520, and it is easily proved that no other order below 5616 affords more than a single type of simple group.

In what follows, the exhaustive enumeration of orders is carried as far as 6232. The only orders found are those of Dickson's table, viz., 4080, 5616, 6048, 6072. There is only one simple group of each of the orders 4080 and 6072; whether there is more than one in the other two cases remains to be decided.

Elementary considerations exclude all orders but the following:

3648	4080	5472	6048
3744	4320	5616	6072
4032	5040	5760	

A bare epitome of the reduction process is here given,—just sufficient to enable the reader to retrace the essential steps. Primitive substitution groups are completely known up to degree 20; an unknown simple group cannot have a set of less than 21 conjugate subgroups. In the text the letters G, H, I, s represent entire group, subgroup, invariant subgroup and element.

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† See this BULLETIN, vol. 5, p. 474; and *Linear Groups*, p. 309.

‡ AMERICAN JOURNAL, vol. 34, p. 361.

§ See this BULLETIN, vol. 28, p. 98.

2. *Order* $3648 = 64 \cdot 3 \cdot 19$. In a simple group of this order the 96 H_{38} would supply 1728 s_{19} and 456 s_2. The group has then 2432 elements whose orders divide $19 \cdot 64$, and only 247 whose orders divide 64 remain to be located. But each of the 456 s_2 above is invariant in an H_8 of type (1 1 1) and containing 4 of the s_2. Each H_{64} contains 2 of these H_8; they generate an H_{16} which has another H_8 with 2 cyclical H_4, whose common H_2 is invariant in H_{64} and has 57 conjugates in the whole group; the 114 conjugate H_4 supply $57 \cdot 5 = 285$ elements where only 247 were permissible.

3. *Order* $3744 = 32 \cdot 9 \cdot 13$. The 144 H_{26} supply 1728 s_{13} and 468, 234 or 156 s_2. The H_{32} are non-abelian. There is not room for 208 independent H_9; an H_3 occurs in 4 H_9 and is invariant in an H_{36}, H_{72} or H_{144}. The first of these cases is quickly rejected. In the second the invariant H_3 of the H_{72} might have 1 or 13 conjugates in the H_{72}, but the latter choice would result in the entire group having only 105 elements whose orders divide 9; there are 208 H_9 and each has 3 H_3 common with other H_9 and 1 H_3 not in any other H_9; the total number of elements of order 3 is then 728, and the number of order 6 is at least $3 \cdot 52 \cdot 6 = 936$. The H_{72} has an invariant H_4; this has conjugates in the H_{72}, which is found to have 9 dihedral H_8 containing 21 s_2 conjugate in the entire group and each common to 3 cyclical H_4; these supply 147 s_2 and s_4. These s_2 are not those of the H_{26}; there must be at least 156 of the latter, but this would lead to at least 312 new s_6 and a total in excess of the order of G.

Hence an H_3 common to two H_9 is invariant in an H_{144} and G is expressible in 26 letters. There are 52 H_9 and each has 2 H_3 in 21 letters and 2 H_3 in 24 letters. The H_{144} has transitive sets of $9 + 12 + 4$ letters; the construction does not succeed, there is no simple group of order 3744.

4. *Order* $4032 = 64 \cdot 9 \cdot 7$. A simple group of this order would be expressible in 21 letters, but not in 64 letters. There are 288 H_7. An H_3 common to two H_9 is common

to four H_9 and is invariant in an H_{36}, H_{72} or H_{144}, all of which cases are successively rejected. There are then 28 independent, non-cyclical H_9, and their 112 H_3 are all conjugate. In the expression of G in 28 letters the H_{144} that leaves one letter fixed has a transitive set of 9 letters; it is one-to-one isomorphic with the unique (doubly) transitive H_{144} in 9 letters; it has 36 s_8 in 26 letters, 54 s_4 in 24 letters, 21 s_2 in 24 letters, 8 s_3 and 24 s_6 each in 27 letters. All but 378 elements of G are now located.

When G is expressed in 21 letters, each H_{192} in 20 letters contains 16 H_3, each in 18 letters, giving 32 s_3 and 32 s_6; there are 3 H_{64}, and the 20 letters divide into transitive sets of $8+12$. The H_{192} has an invariant I_4 leaving 9 letters fixed and having 9 conjugates in its I_{32}, whose elements other than identity are all of order 2. The construction fails; there is no simple group of order 4032.

5. *Order* $4080 = 16 \cdot 3 \cdot 5 \cdot 17$. There are 120 H_{34}, supplying 1920 s_{17} and 255 or 510 s_2. If there are 51 H_{80}, each of them has 5 H_{16}; there are 85 or 340 H_3, each with a train of s_6; the case does not succeed. There are therefore 136 H_{30}; their H_{15} furnish $136 \cdot 14 = 1904$ elements; the group has exactly 255 s_2, which make up 17 independent H_{16}. The only simple group of order 4080 is the well known $LF(2, 2^4)$.

6. *Order* $4320 = 32 \cdot 27 \cdot 5$. Here there must be an H_9 invariant in an H_{108} with 4 H_{27} and 9 H_4 each invariant in an H_{12} of the H_{108}. This H_{12} has a single H_4 whose s_2 are conjugate; the 9 H_4 are independent, the elements of the H_{108} are all accounted for and they do not include any s_6. The construction does not succeed, there is no simple group of this order.

7. *Order* $5040 = 16 \cdot 9 \cdot 5 \cdot 7$. The 120 H_{42} are of the seven-letter type. There are 126 H_{40}, each with 5 H_8 with invariant I_2. The case where I_2 has 21 conjugates in G does not work through; I_2 has 126 conjugates and each H_{40} contains 6 or 26 of them and in fact just 6 of them.

The H_8 of the H_{40} are of type (2 1) or (1 1 1), and each of them contains just one s_2 of the H_{42} above. The number of these s_2 in the entire group is 105; each is invariant in an H_{48} containing 6 of the I_2 above and 3 of the H_8 above. A detailed study of the H_{48} shows that a simple group of order 5040 cannot be fitted together.

8. *Order* $5472 = 32 \cdot 9 \cdot 19$. The 96 non-cyclical H_{57} contain $16 \cdot 19$ or $32 \cdot 19\, H_3$. A different H_3 is common to 4 H_9 and invariant in an H_{72}; the group being expressed in 76 letters, the invariant H_3 of the H_{72} affects 75 letters, there are in all 304 H_9, each has a second H_3 common to 4 H_9 and affecting 72 letters; the remaining H_3 affect 75 letters. The H_{72} has an invariant I_4, with conjugates in H_{72}; hence its H_8 are dihedral or of type (1 1 1), and it turns out that there are 3 of them and their type is (1 1 1). An s_2 of their common I_4 must have just 12 conjugates in H_{72}, but this proves impracticable and there is no simple group of order 5472.

9. *Order* $5760 = 128 \cdot 9 \cdot 5$. Here an H_{32} is common to 3 H_{64} and invariant in an H_{192}; G is expressible in 30 letters. The 576 non-cyclic H_{10} supply 2304 s_5. There are 640, 160 or 40 H_9; in any case an H_3 is common to 4 H_9 and invariant in an H_{72} or H_{144}. The H_{72} must have 3 H_8, dihedral or (1 1 1), neither case works through. The group is then expressible in 40 letters; its H_3 are not all conjugate, this case fails on comparing with the expression in 30 letters. The H_{144} in 39 letters must have an invariant H_3 in 39 letters, 4 conjugate H_3 in 36 letters, and a residue of 8 H_3 in 39 letters. Its transitive systems are $9 + 12 + 18$ or $9 + 12 + 9 + 9$. An examination of the transitive set of 12 letters discloses the impossibility of a simple group of this order.

10. *Order* $6072 = 8 \cdot 3 \cdot 11 \cdot 23$. The 24 H_{253} furnish 276 H_{11} each invariant in a non-cyclical H_{22} whose s_2 affect 24 letters. Only one construction is possible; there is only one type of simple group of this order.

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