A CONVERGENCE PROOF FOR SIMPLE AND MULTIPLE FOURIER SERIES*

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The purpose of this paper is to establish the convergence of both the simple and the multiple Fourier series with a second order linear homogeneous differential equation as a starting point. The method of proof for the simple series is essentially that of Birkhoff† applied to a special case. In the extension of the theory to the multiple series, the argument is similar to that used by Camp‡ in connection with a first order equation. Apart from any relation with more general theory, the proofs given here are of interest because of the elegance with which they lead to important results.

THEOREM I. *Let $f(x)$ be made up of a finite number of pieces in the interval $-\pi \leqq x \leqq \pi$, each real, continuous, and with a continuous derivative. For $-\pi < x < \pi$ the Fourier series for $f(x)$ converges to $\frac{1}{2}[f(x-0)+f(x+0)]$. For $x = \pm\pi$ it converges to $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$.*

We start with the differential equation

$$(1) \qquad y'' + \varrho^2 y = 0,$$

and the boundary conditions

$$(2) \qquad y(-\pi) = y(\pi), \qquad y'(-\pi) = y'(\pi),$$

where x is the independent variable and ϱ^2 is a parameter.

It is easily seen that a necessary and sufficient condition that the system (1), (2) have a solution is that ϱ be a positive or negative integer, or zero. These integral values of ϱ

* The means of approach to Fourier series which is employed in this paper was suggested to a class of graduate students by Professor R. D. Carmichael; the problem was solved completely or in part by several members of the group.

† TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), p. 390.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 123–34.

are called the *characteristic values.* We now seek a function which, for all values of ϱ other than the characteristic values, satisfies conditions (2) and comes as near as possible to satisfying (1) throughout the interval $-\pi \leqq x \leqq \pi$. Under one interpretation of "as near as possible" this function, which we shall call $G(x, \xi; \varrho^2)$, is the function which satisfies the boundary conditions and also satisfies equation (1) at every point of the interval, $-\pi \leqq x \leqq \pi$, except at one point ξ, at which point $G(x, \xi; \varrho^2)$ itself is continuous but its derivative with respect to x has a jump of 1. We may compute $G(x, \xi; \varrho^2)$ as follows: let

$$G(x, \xi; \varrho^2) = \begin{cases} c_1 \cos \varrho x + c_2 \sin \varrho x, & \text{when } -\pi \leqq x < \xi, \\ c_3 \cos \varrho x + c_4 \sin \varrho x, & \text{when } \xi < x \leqq \pi, \end{cases}$$

where the c's are independent of x. There are four relations among the c's, which are sufficient to determine them uniquely; two imposed by the boundary conditions, one by the condition of continuity of $G(x, \xi; \varrho^2)$ at $x = \xi$, and the fourth by the condition of discontinuity of $G'_x(x, \xi; \varrho^2)$ at $x = \xi$. This last relation, written explicitly, is

$$G'_x(\xi + 0, \xi; \varrho^2) - G'_x(\xi - 0, \xi; \varrho^2) = 1,$$

or

$$-c_3 \varrho \sin \varrho \xi + c_4 \varrho \cos \varrho \xi - (-c_1 \varrho \sin \varrho \xi + c_2 \varrho \cos \varrho \xi) = 1.$$

In this way we find

$$(3)\quad G(x, \xi; \varrho^2) = \frac{1}{2\varrho}\left[\operatorname{sgn}(x - \xi) \sin(\varrho x - \varrho \xi) + \frac{\cos \varrho \pi \cos(\varrho x - \varrho \xi)}{\sin \varrho \pi}\right].$$

Considered as a function of ϱ, $G(x, \xi; \varrho^2)$ is analytic except at the characteristic values of ϱ. For ϱ a positive or negative integer, $G(x, \xi; \varrho^2)$ has a pole of the first order, while at $\varrho = 0$ it has a pole of the second order. In order to reduce this second order pole to one of the first order, multiply both members of (3) by ϱ and then consider the function $\varrho G(x, \xi; \varrho^2)$, which is analytic except for simple poles at the characteristic values of ϱ. Denoting by $R_n(x, \xi)$ the residue of $\varrho G(x, \xi; \varrho^2)$ at $\varrho = n$, one readily finds that

$$R_n(x,\xi) = R_{-n}(x,\xi) = \frac{\cos(nx - n\xi)}{2\pi},$$

whence

$$(4) \qquad \frac{1}{2\pi i}\int_{\Gamma_n} \varrho\, G(x,\xi;\varrho^2)\, d\varrho = \frac{1}{2\pi}(\cos nx \cos n\xi + \sin nx \sin n\xi),$$

where Γ_n is a circle in the ϱ-plane with center at $\varrho = n$ and radius $\frac{1}{2}$. The Fourier series for $f(x)$ is

$$(5) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$(6) \qquad \begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos n\xi f(\xi)\, d\xi, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\xi f(\xi)\, d\xi, \qquad (n = 0, 1, 2, \ldots). \end{aligned}$$

The general term of this series is

$$a_n \cos nx + b_n \sin nx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos nx \cos n\xi + \sin nx \sin n\xi) f(\xi)\, d\xi,$$

which, by virtue of (4), may be written in the form

$$a_n \cos nx + b_n \sin nx = \frac{1}{\pi i}\int_{\Gamma_n}\int_{-\pi}^{\pi} \varrho\, G(x, \xi; \varrho^2)\, f(\xi)\, d\xi\, d\varrho.$$

This same term will clearly be obtained if the contour integration is along Γ_{-n} instead of Γ_n. Hence, if C_k is a circle in the ϱ-plane with center at the origin and radius $k + \frac{1}{2}$, and if S_k denotes the sum of the first $k+1$ terms of (5), we have

$$S_k = \frac{1}{2\pi i}\int_{C_k}\int_{-\pi}^{\pi} \varrho\, G(x,\xi;\varrho^2) f(\xi)\, d\xi\, d\varrho.$$

The limit of this double integral is desired as k becomes infinite. For this purpose the integral is broken up into two parts, $I_k^{-\pi,x}$ and $I_k^{x,\pi}$, the limits of ξ being $-\pi$ and x in $I_k^{-\pi,x}$ and x and π in $I_k^{x,\pi}$, where $-\pi < x < \pi$.

Considering $I_k^{-\pi,x}$ and integrating by parts with respect to ξ, we have

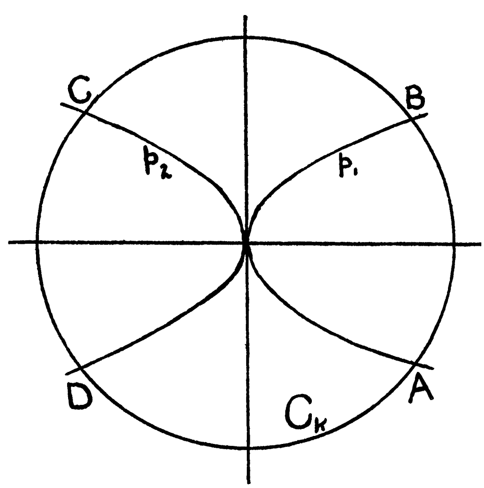
$$I_k^{-\pi,x} = \frac{1}{4\pi i}\int_{C_k}\int_{-\pi}^{x}\frac{\cos\varrho(\pi - x + \xi)}{\sin\varrho\pi} f(\xi)\,d\xi\,d\varrho$$

$$= \frac{1}{4\pi i}\int_{C_k}\left[\frac{1}{\varrho}f(x-0) + \frac{\sin\varrho x}{\varrho\sin\varrho\pi}f(-\pi+0) - \frac{1}{\varrho}\int_{-\pi}^{x}\frac{\sin\varrho(\pi - x+\xi)}{\sin\varrho\pi}f'(\xi)\,d\xi\right]d\varrho.$$

As k becomes infinite each of the three parts of $I_k^{-\pi,x}$ contributes a certain amount to the sum of the series which we are seeking. The contribution of

$$\frac{1}{4\pi i}\int_{C_k}\frac{1}{\varrho}f(x-0)\,d\varrho$$

is easily seen to be $\frac{1}{2}f(x-0)$. We now show that the contribution of each of the other two parts is zero. In the ϱ-plane let p_1 and p_2 be two parabolas with vertices at the origin and axes coinciding with the axis of reals, p_1 lying in a right half-plane, meeting C_k in the points A and B, and p_2 lying in a left half-plane, meeting C_k in the points C and D. Then along the arcs AB and CD $|\sin\varrho x/\sin\varrho\pi|$ is bounded while the product of $|1/\varrho|$ and the length of the path of integration goes to zero as k becomes infinite. On the other hand, along



the arcs BC and DA $\lim_{k=\infty} | \sin \varrho x / \sin \varrho \pi | = 0$, since the imaginary part of ϱ becomes infinite with k, while the product of $| 1/\varrho |$ and the length of the path of integration is bounded. Therefore the contribution of the second part of $I_k^{-\pi,x}$ is zero. To see that the third part of $I_k^{-\pi,x}$ also contributes zero, take the absolute value of the integrand, replace $|f'(\xi)|$ by an upper bound M, integrate first with respect to ξ and then, breaking up C_k with the parabolas as before, integrate along C_k.

In precisely the same way one sees that as k becomes infinite the total contribution of $I_k^{x,\pi}$ is $\frac{1}{2}f(x+0)$. The theorem is now proved for interior points of the interval. It may be extended to include the end points $+\pi$ and $-\pi$ in the following way. Let $\Phi(x)$ be a function of period 2π, identical with $f(x)$ in the interval $-\pi \leqq x < \pi$. By the part of the theorem already proved $\Phi(x)$ is represented by its Fourier series in the interval $0 \leqq x \leqq 2\pi$ except possibly at $x = \pi$. Let $g(x) = \Phi(x+\pi)$. It may easily be shown that the Fourier constants of $g(x)$ as calculated directly from formulae (6) are identical with those obtained by replacing x by $x+\pi$ in the Fourier series for $\Phi(x)$. But since $g(x)$ satisfies all the conditions of the theorem, its Fourier series converges at $x = 0$ to $\frac{1}{2}[g(+0)+g(-0)]$. Hence at $x = \pi$, the Fourier series for $f(x)$ converges to

$$\begin{aligned}\tfrac{1}{2}[g(+0)+g(-0)] &= \tfrac{1}{2}[\Phi(\pi+0)+\Phi(\pi-0)] \\ &= \tfrac{1}{2}[f(-\pi+0)+f(\pi-0)].\end{aligned}$$

Similarly at $x = -\pi$, the Fourier series for $f(x)$ converges to the same value.

THEOREM II. *Let $f(x_1, x_2, \ldots, x_k)$ be made up in the region $-\pi \leqq x_j \leqq \pi$ $(j = 1, 2, \ldots, k)$ of a finite number of pieces, each real, continuous, and with continuous partial derivatives with respect to $x_1, x_2, \ldots, x_k$. Then the Fourier series for $f(x_1, x_2, \ldots, x_k)$ converges to the value $\frac{1}{2^k}\sum f(x_1 \pm 0, x_2 \pm 0, \ldots, x_k \pm 0)$, where the summation sign means the sum of the 2^k terms obtained by taking all possible combinations of the positive and negative signs. If $x_j = \pm\pi$,* *instead of* x_j+0 *and* x_j-0 *read* $-\pi+0$ *and* $\pi-0$ *respectively.*

For the sake of simplicity the proof is given here for $k=2$. The method is clearly applicable to the general case.

We now use the system

$$\left.\begin{array}{l} y_j''+\varrho_j^2 y=0, \\ y_j(-\pi)=y_j(\pi), \qquad y_j'(-\pi)=y_j'(\pi), \end{array}\right\} (j=1,2).$$

Calculating the residue of $\varrho_1 G(x_1, \xi_1; \varrho_1^2)$ at $\varrho_1=m$, and that of $\varrho_2 G(x_2, \xi_2; \varrho_2^2)$ at $\varrho_2=n$ and taking the product of these residues, one finds

$$\begin{aligned} (7)\qquad & \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_n'}\int_{\Gamma_m} \varrho_1 G(x_1, \xi_1; \varrho_1^2)\varrho_2 G(x_2, \xi_2; \varrho_2^2)\, d\varrho_1 d\varrho_2 \\ & = \frac{1}{4\pi^2}[\cos mx_1 \cos nx_2 \cos m\xi_1 \cos n\xi_2 \\ & + \cos mx_1 \sin nx_2 \cos m\xi_1 \sin n\xi_2 + \sin mx_1 \cos nx_2 \sin m\xi_1 \cos n\xi_2 \\ & + \sin mx_1 \sin nx_2 \sin m\xi_1 \sin n\xi_2], \end{aligned}$$

where Γ_m is a circle in the ϱ_1-plane about the pole $\varrho_1=m$ and Γ_n' is a circle in the ϱ_2-plane about the pole $\varrho_2=n$.

The Fourier series for $f(x_1, x_2)$ is

$$\begin{aligned} (8)\qquad & \sum_{m,n=0}^{\infty} \varepsilon_{mn}[A_{mn}\cos mx_1 \cos nx_2 + B_{mn}\cos mx_1 \sin nx_2 \\ & + C_{mn}\sin mx_1 \cos nx_2 + D_{mn} \sin mx_1 \sin nx_2], \end{aligned}$$

where

$$\varepsilon_{mn} = \begin{cases} \frac{1}{4}, \text{ if } m=n=0, \\ \frac{1}{2}, \text{ if } m=0,\ n>0, \text{ or if } m>0,\ n=0, \\ 1, \text{ if } m>0,\ n>0, \end{cases}$$

and where

$$(9)\qquad \begin{aligned} A_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos m\xi_1 \cos n\xi_2 f(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \\ B_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos m\xi_1 \sin n\xi_2 f(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \\ C_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sin m\xi_1 \cos n\xi_2 f(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \\ D_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sin m\xi_1 \sin n\xi_2 f(\xi_1, \xi_2)\, d\xi_1 d\xi_2. \end{aligned}$$

Combining formulas (7) and (9), we have

$$\begin{aligned}(10)\qquad & A_{mn}\cos mx_1\cos nx_2+B_{mn}\cos mx_1\sin nx_2\\ &\quad +C_{mn}\sin mx_1\cos nx_2+D_{mn}\sin mx_1\sin nx_2\\ &=\left(\frac{1}{\pi i}\right)^2\int_{\Gamma_n'}\int_{\Gamma_m}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varrho_1 G(x_1,\xi_1;\varrho_1^2)\,\varrho_2 G(x_2,\xi_2;\varrho_2^2)\\ &\qquad f(\xi_1,\xi_2)\,d\xi_1 d\xi_2 d\varrho_1 d\varrho_2\,.\end{aligned}$$

The first member of this equation, which is the general term of (8), will also be obtained if Γ_m is replaced by Γ_{-m} or Γ_n' by Γ_{-n}', or if both these changes are made simultaneously. Hence, if Γ_m and Γ_n' are replaced by C_μ and C_ν' respectively, where C_μ is a circle in the ϱ_1-plane with center at the origin and radius $\mu+\frac{1}{2}$, and C_ν' is a circle in the ϱ_2-plane with center at the origin and radius $\nu+\frac{1}{2}$, the second member of (10) gives four times the sum of the terms of (8) for which $m\leqq\mu$ and $n\leqq\nu$. Let $S_{\mu\nu}$ be this sum. Then we have

$$\begin{aligned}S_{\mu\nu}=\left(\frac{1}{2\pi i}\right)^2\int_{C_\nu'}\int_{C_\mu}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varrho_1 G(x_1,\xi_1;\varrho_1^2)\,\varrho_2 G(x_2,\xi_2;\varrho_2^2)\\ f(\xi_1,\xi_2)\,d\xi_1 d\xi_2 d\varrho_1 d\varrho_2\,.\end{aligned}$$

To find the limit of this four-fold integral as μ and ν become infinite, theorem I may be used, for by interchanging the order of the integrations with respect to ξ_2 and ϱ_1 the integral becomes simply a succession of two double integrals, each of the type evaluated in the proof of Theorem I. Integrating first with respect to ξ_1 and ϱ_1, and then with respect to ξ_2 and ϱ_2, we have

$$\begin{aligned}\lim_{\mu,\nu=\infty} S_{\mu\nu}=\tfrac{1}{4}[f(x_1-0,x_2-0)+f(x_1+0,x_2-0)\\ +f(x_1-0,x_2+0)+f(x_1+0,x_2+0)].\end{aligned}$$

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