nomials in y, viz. $P_1^{(k)}$, $P_2^{(k)}$, ..., $P_N^{(k)}$ such that $f_1^{(k)} : f_2^{(k)} : \cdots$
$: f_N^{(k)} = P_1^{(k)} : P_2^{(k)} : \cdots : P_N^{(k)}$, there will exist such polynomials
in which the lowest powers of y are respectively equal to
the lowest powers of y in the series $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$. Hence
there will exist a series $\Phi^{(0)}(y)$ with non-vanishing constant
term satisfying (6). It is then easy to construct the poly-
nomial $P^{(k)}$ satisfying the relation $\Phi^{(0)} f^{(k)} = P^{(k)}$. In view
of the homogeneity of f this relation may be written in the
form $\Phi f = P$. From this may be obtained the equivalent
form (1).

If $f(y; x_1, \ldots, x_n)$ is not homogeneous in $x_1, x_2, \ldots, x_n$
the condition of the theorem is *not sufficient*.

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THE CLASS NUMBER RELATIONS IMPLICIT IN THE DISQUISITIONES ARITHMETICAE*

BY E. T. BELL

1. *Introduction.* The point of this note is its moral,
which is to the effect that in arithmetic attention to trifles
sometimes leads to beautiful and recondite truths. In
particular, certain important expansions of Kronecker and
Hermite relating to the number $F(n)$ of uneven classes of
binary quadratic forms of negative determinant $-n$ are
implicit in § 292 of the *Disquisitiones Arithmeticae* of Gauss,
and might have been read off from there at a glance by
anyone familiar with the *Fundamenta Nova* of Jacobi,
thirty years before Kronecker first came upon them by
the devious route of complex multiplication. The relevant
trifle in this instance is changing the sign of an arbitrary
constant throughout an algebraic identity.

It is well known that the tenth of Kronecker's relations[*] is equivalent to his expansion

$$\text{(XI)} \quad 12 \sum_0^\infty E(n)\, q^n = \left[\sum_{-\infty}^\infty q^{n^2} \right]^3 = \vartheta_3^3(q),\; E(n) = F(n) - F_1(n),$$

where $F_1(n)$ is the number of even classes, which is merely the algebraic expression of the theorem of Gauss (loc. cit.) on the number of representations of n as a sum of three squares; and it has recently been remarked by Mordell[†] that Hermite's important formula

$$\text{(1)} \quad \sum_0^\infty F(8n+3)\, q^{8n+3} = (q + q^9 + q^{25} + \cdots)^3 = \tfrac{1}{8}\vartheta_2^3(q^4),$$

and those of Kronecker (changed from k, K to ϑ notation),

$$\text{(2)} \quad 4 \sum_0^\infty F(4n+2)\, q^{4n+2} = \vartheta_3(q^4)\, \vartheta_2^2(q^4),$$

$$\text{(3)} \quad 4 \sum_0^\infty F(4n+1)\, q^{4n+1} = \vartheta_2(q^4)\, \vartheta_3^2(q^4),$$

are implicit in (XI) and can be obtained from it by elementary algebra. Note in passing that (1) is simply the algebraic translation of § 293 of the *Disquisitiones Arithmeticae*. Both Kronecker and Hermite made much use of (1), (2), (3) in their successive simplification of the classical relations and in further deductions, and these identities reappear repeatedly in works of later writers. It.is interesting therefore to see that (1), (2), (3) follow from (XI) even more simply than as shown by Mordell; in fact they result from (XI) by changing the sign of q. That is, (XI), (1), (2), (3) are immediate consequences of Gauss' theorem. The deduction of class number relations from these four identities by combining them with the expansions of Jacobi is not only classical but obvious, and it seems surprising that the immediate step of translating the application which Gauss made of his theorem, to representations as

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[*] Cf. Dickson's *History*, vol. 3, pp. 108, 109, 113.
[†] MESSENGER OF MATHEMATICS, vol. 45 (1915) pp. 78, 79.

sums of four squares, into the equivalent class number relation was not taken much earlier than it was.

2. *Consequences of Gauss' Theorem.* In (XI) change q into $-q$:

(XI$_1$) $\quad 12\sum_{0}^{\infty}(-1)^n\, E(n)q^n = \left[\sum_{-\infty}^{\infty}(-1)^n\, q^{n^2}\right]^3 = \vartheta_0^3(q).$

Let $n_1, n_2, \ldots$ denote arbitrary integers $\gtreqless 0$, $m_1, m_2, \ldots$ odd integers $\geqq 0$, and write $N(n = x^2 + y^2 + z^2)$ for the number of integer solutions $x, y, z \gtreqless 0$ of $n = x^2 + y^2 + z^2$. Equate coefficients of q^n in (XI$_1$):

$$12(-1)^n E(n)$$
$$= N(n = 4n_1^2 + 4n_2^2 + 4n_3^2) + N(n = 4n_1^2 + m_2^2 + m_3^2)$$
$$+ N(n = m_1^2 + 4n_2^2 + m_3^2) + N(n = m_1^2 + m_2^2 + 4n_3^2)$$
$$- N(n = m_1^2 + m_2^2 + m_3^2) - N(n = m_1^2 + 4n_2^2 + 4n_3^2)$$
$$- N(n = 4n_1^2 + m_2^2 + 4n_3^2) - N(n = 4n_1^2 + 4n_2^2 + m_3^2),$$
$$= N(n = 4n_1^2 + 4n_2^2 + 4n_3^2) + 3N(n = 4n_1^2 + m_2^2 + m_3^2)$$
$$- N(n = m_1^2 + m_2^2 + m_3^2) - 3N(n = m_1^2 + 4n_2^2 + 4n_3^2).$$

Since
$$4n_1^2 + 4n_2^2 + 4n_3^2 \equiv 0 \mod 4,$$
$$4n_1^2 + m_2^2 + m_3^2 \equiv 2 \mod 4,$$
$$m_1^2 + m_2^2 + m_3^2 \equiv 3 \mod 8,$$
$$m_1^2 + 4n_2^2 + 4n_3^2 \equiv 1 \mod 4,$$

we have $12E(n) = N(4n = 4n_1^2 + 4n_2^2 + 4n_3^2)$, which is merely (XI) again, and

$$N(8n + 3 = m_1^2 + m_2^2 + m_3^2) = 12E(8n + 3),$$
$$N(4n + 2 = 4n_1^2 + m_2^2 + m_3^2) = \ \ 4E(4n + 2),$$
$$N(4n + 1 = m_1^2 + 4n_2^2 + 4n_3^2) = \ \ 4E(4n + 1);$$

which are equivalent respectively to

$$\left(\sum q^{m_1^2}\right)^3 = 12\sum q^{8n+3}E(8n + 3),$$
$$\sum q^{4n_1^2} \times \left(\sum q^{m_1^2}\right)^2 = \ \ 4\sum q^{4n+2}E(4n + 2),$$
$$\sum q^{m_1^2} \times \left(\sum q^{4n_1^2}\right)^2 = \ \ 4\sum q^{4n+1}E(4n + 1).$$

But these are identical respectively with (1), (2), (3), since $E(8n + 3) = \frac{2}{3}F(8n + 3)$, $E(4n + 2) = F(4n + 2)$,
$$E(4n + 1) = F(4n + 1).$$

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