

## NATIONAL RESEARCH COUNCIL REPORT ON ALGEBRAIC NUMBERS

*Algebraic Numbers. Report of the Committee on Algebraic Numbers, National Research Council.* By L. E. Dickson, H. H. Mitchell, H. S. Vandiver, G. E. Wahlin. BULLETIN OF THE NATIONAL RESEARCH COUNCIL, vol. 5, part 3, number 28, February, 1923. Washington, D. C. 96 pp.

Nowhere is the craft and subtlety of the mathematician more ingeniously developed than in the theory of algebraic numbers. The subject is dotted over with gins and snares, not to say boobytraps, and more than one reputable arithmetician has fallen foul of its pitfalls, thereby exhibiting himself as a warning to the initiated who have sense enough to heed his misfortunes. The unwary, with all the courage of their kind, rush in and are quickly slain. But for those who are gifted with caution and perseverance the theory has rich rewards. As an emancipator of the intellect the theory of ideals is at least the peer of non-euclidean geometry, yet it is but little cultivated by professional mathematicians and is almost wholly unappreciated by mathematical philosophers. The reason is not far to seek: even schoolboys are easily if superficially impressed by the spectacular heterodoxies of geometry, while hard labor and a mature judgment are prerequisites for the mere apperception of the quieter and perhaps more profound beauties of modern arithmetic. Should some mathematician with the necessary talents popularize this subject as non-euclidean geometry has been popularized, he would do an important service to both mathematics and philosophy.

The Report by Dickson, Mitchell, Vandiver, and Wahlin has the twofold object of bringing up to date Hilbert's classic *Bericht* of 1894-95 on the theory of algebraic number fields, and of supplementing the Bericht by accounts of the literature on fields of functions and related topics which Hilbert omitted. It is stated in the preface that Hilbert's report and the present one exhaust the literature. So far as the reviewer can judge this claim is substantiated. There are a few printer's errors, but none that will cause any inconvenience.

This Report differs in one essential respect from both Hilbert's Bericht and from the Bulletins on mathematical physics issued by the National Research Council. It would not be possible to gain a working knowledge of the topics discussed from the extremely condensed statements in the 96 pages of the Report. Nevertheless this Report is admirably fitted to the use of the experts for whom it doubtless is intended.

The responsibility for the several main divisions of the Report is as follows: quadratic, galois and abelian fields, units in a general field,

(MITCHELL); cubic fields, norms and congruences in a general algebraic field, classes of ideals, Klassenkörper, complex multiplication of elliptic functions, higher reciprocity laws and Kummer fields, (VANDIVER); distribution of prime ideals, miscellaneous topics in algebraic numbers, cyclotomy, fields of functions, (DICKSON); Hensel's  $p$ -adic numbers (WAHLIN). Remarkable uniformity of treatment has been maintained throughout, the style following closely that of Dickson's *History of the Theory of Numbers*. In practically all instances the chief results of the memoir abstracted are concisely stated; there are also occasional indications of the methods followed or short summaries of the proofs. A particularly valuable feature is the correction of such errors in the literature as have been noted by the authors, each of whom is an expert in the topics which he discusses.

Although it would be absurd to attempt a précis of this work which is itself a concise summary and which carries condensation to the limit, it may not be out of place here to note one or two points of general interest, particularly for the encouragement of beginners in this extensive and difficult province of modern arithmetic. First, the relatively large amount of space devoted to cyclotomy is proof that this venerable subject—it was born with Gauss' discovery of 1796—is still lusty. Indeed several of the reports hint at unexhausted lodes where a novice might yet dig out nuggets of value. Quite recently, for example, Burnside has considered octosection. These smaller finds are valuable chiefly as whetters of the appetite for discovery; the mother lode lies elsewhere. Again, the fascinating highway of Hensel's  $p$ -adic numbers, but lately opened up to general traffic and bearing as it does toward analysis, offers an attractive and less hazardous entrance to the entire theory than do the older, strictly arithmetic approaches. The literature of this subject has a refreshing newness; it has not yet grown dishearteningly vast, nor is it overloaded with minute and harrassing technicalities. To reach quickly the heart of the classical theory by well travelled roads one cannot do better than follow Landau's exposition (Teubner, 1918), as yet unsurpassed for brevity and clearness.

Although, as can be seen from an inspection of this Report and of current literature, the theory of algebraic numbers owes most of its extraordinarily rapid development to the German school, evidence is not lacking that at last the mathematical public in America, England, and France is becoming aware of the splendid creations of Kummer, Dedekind, Kronecker, Hensel and their followers. The contributions of American mathematicians have been few but choice. To cite only three, all mentioned in the Report, we may recall the fundamental and far reaching researches of E. H. Moore and Dickson on the classes of residues of a prime ideal modulus, Mitchell's correction of errors in the work of Kummer which escaped the notice of so penetrating a mind

as H. J. S. Smith, and Vandiver's numerous contributions to the same end. Last there is the recent work of Dickson, which bids fair to be epoch making, on *Algebras and their Arithmetics*,\* where the classic theory of algebraic numbers finds a simple and profound generalization.

With the books of Landau and Dickson, the report of Hilbert and that of the present authors now available, it is to be hoped that algebraic numbers, one of the major divisions of modern mathematics, will not much longer remain in learned obscurity, but will take its rightful place as one of the chief glories of any liberal mathematical education.

E. T. BELL

### VOLUMES II AND III OF DICKSON'S HISTORY

*History of the Theory of Numbers.* By L. E. Dickson. Vol. II, *Diophantine Analysis*, xxv + 803 pp. Vol. III, *Quadratic and Higher Forms*, (with a chapter on the class number by G. H. Cresse). iv + 313 pp. The Carnegie Institution, Washington, 1923.

Since the time of Gauss, the theory of numbers has developed in a number of different directions. Let us examine this development prior to the year 1890. Dirichlet and Riemann founded the analytic prime number theory; Kummer, Kronecker, and Dedekind created the theory of algebraic numbers; Eisenstein, Hermite, Smith, and Minkowski developed the arithmetic theory of forms; Jacobi, Eisenstein, Kronecker, Smith, and Hermite applied the theory of elliptic functions to various problems. It will be noted that, in the main, this progress centered about a few great names. The discoveries of these men did not excite the attention of other mathematicians in many cases because the contents of the original papers were often complicated and difficult to read, and few suitable texts were provided to meet the needs of the beginner.

In considering the period between 1890 and 1900, however, a decided change is noted. In this interval appeared the *Lehrbuch der Algebra* of Weber and Hilbert's *Bericht über die Theorie der Algebraischen Zahlen*. These works and the original papers of the same authors appear to have exercised a profound influence on a number of able young mathematicians. In another line, Hadamand and de la Vallée Poussin obtained epoch making results in the theory of the Riemann zeta function, with applications to the asymptotic distribution of prime numbers. Minkowski founded a geometry of numbers which has bearing on many parts of the number theory. Dickson initiated his extensive contributions to the subject by developing the theory of finite fields,

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\* This sentence was written by the reviewer before the award to Professor Dickson of the Cincinnati Prize. See page 90 of this issue.  
THE EDITORS.