BROUWER'S CONTRIBUTIONS TO THE FOUNDATIONS OF MATHEMATICS*

BY ARNOLD DRESDEN

1. *Introduction.* In a number of papers, published from 1907 on, Professor L. E. J. Brouwer, of the University of Amsterdam, has developed ideas which affect the foundations of mathematics in a fundamental way. While some of his papers are readily available to American mathematicians,† there are several others which are less accessible. On account of its critique of some of our most fundamental concepts and methods, the position of Brouwer may have a far reaching effect upon the future development of mathematics. In his *Begründung der Mengenlehre*, he has made a beginning with a revision of a basic field of modern mathematics in accordance with his point of view. But, whatever their ultimate significance may be, the conclusions which Brouwer reaches are certainly interesting. Moreover, they are indispensable as a background for an appreciation of his *Begründung der Mengenlehre*, as well as for understanding the controversial discussions on the foundations of mathematics of Weyl and Hilbert.‡

For this reason, it has seemed worth while to present Brouwer's most important ideas concerning the foundations of mathematics to American readers. For this purpose, we

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† See *Intuitionism and formalism*, this BULLETIN, vol. 20 (1913), p. 81; Review of Schoenflies-Hahn, *Die Entwickelung der Mengenlehre*, JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), p. 78; *Intuitionistische Mengenlehre*, JAHRESBERICHT DER VEREINIGUNG, vol. 28 (1920), p. 203; *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten*, Amsterdam, 1918–19.

‡ See Weyl, *Über die neue Grundlagenkrise der Mathematik*, MATHEMATISCHE ZEITSCHRIFT, vol. 10 (1921), p. 39; Hilbert, *Neubegründung der Mathematik*, ABHANDLUNGEN DER HAMBURGISCHEN UNIVERSITÄT, vol. 1 (1922), p. 157, and *Die logischen Grundlagen der Mathematik*, MATHEMATISCHE ANNALEN, vol. 88 (1922), p. 151.

have used, besides the material referred to above, his book on the foundations of mathematics, (*Over de Grondslagen der Wiskunde,* Amsterdam, Maas & van Suchtelen, 1907) and his article on the unreliability of logical principles (*De onbetrouwbaarheid der logische principes*, TIJDSCHRIFT VOOR WIJSBEGEERTE, vol. 1 (1908)). Our discussion falls into three parts, viz., mathematics and experience, mathematics and mathematical language, mathematics and logic.

2. *Mathematics and Experience.* Brouwer conceives of mathematical thinking as a process of construction, which builds its own universe, independent of the universe of our experience, somewhat as a free design, under the control of nothing but arbitrary choice, restricted only in so far as it is based upon the fundamental mathematical intuition. This intuition, upon which not only mathematical thinking, but all intellectual activity is held to be based, is found in the abstract substratum of all observation of change, "a fusion of continuous and discrete, a possibility of conceiving simultaneously several units, connected by a '*between*' that cannot be exhausted by the interpolation of new units."

It is not to be expected that all mathematicians will agree with this point of view. It is in this conception of the source and of the character of mathematical thinking that the ideas of Brouwer have their root. Its relation to the thought of Plato* and of other Greek philosophers, interesting as it is, must be left untouched here except for the observation that the acceptance of this union of discrete and continuous as the rock bottom of mathematical thinking disposes of the paradoxes of Zeno and of the conflicts of Parmenides somewhat as the theory of relativity disposed of the drag of the ether.†

In a similar way it disposes of many questions in point set theory, which have occupied the attention of mathematicians. For, by combining continuous and discrete in one fundamental concept, it renders futile all attempts at building up one of

* See, e. g., Brunschvicg, *Les Étapes de la Philosophie Mathématique*, p. 49 et seq.

† Compare Brunschvicg, loc. cit., p. 155.

these by means of the other, conceived as independent of the first. But these matters have been dealt with extensively in *Intuitionism and Formalism.*

The fundamental intuitive concept of mathematics must not be thought of as in the nature of an undefined idea, such as occur in postulational theories, but rather as something in terms of which all undefined ideas which occur in the various mathematical systems are to be intuitively conceived, if they are indeed to serve in mathematical thinking. It manifests itself in the intuition of time, which makes possible "repetition, as being object in time and again object."

The position of Brouwer on this point is directly opposed to those of Kant and of Russell, who hold that mathematical thinking cannot be based on the one-dimensional time continuum alone, but that it requires also three-dimensional euclidean space (Kant, *Transcendental Ethics*), or projective space (Russell, *Foundations of Geometry*).

In the first chapter of *Over de Grondslagen der Wiskunde,* Brouwer constructs on the basis of this fundamental intuition the order types ω and η, and the elementary propositions of algebra and geometry. In these building processes, experience plays no part, and Brouwer holds that "in this constructive process, bound by the obligation to notice with care which theses are acceptable to the intuition and which are not, the only possible foundation for mathematics is to be looked for."*

It must of course be remembered that these statements concerning the role of experience are to be taken in the philosophical sense, not in the historical sense. For no one could deny that in the historical development of mathematics, experience played a permanent part. On the other hand, one will have to admit that, while "experimental *science* is linked up with mathematics, experience can never force the

* It may be of interest to compare with this statement, the one found on p. 50 of Boutroux, *L'Idéal Scientifique des Mathématiciens,* in a discussion of the Hellenic conception: "And if it frequently happens that we make mistakes, it is because we have obscured our vision by insufficient exercise of our intuitive faculty."

choice of a *particular* mathematical system." The question as to the role of experience in the development of mathematics seems to the present author still to have a good many aspects that call for further study. Enough has been said however to indicate Brouwer's fundamental thesis and to discuss some of its important consequences.

Two other points need however still to be made clear. It must already have become evident to the reader that Brouwer is not seeking to build up a system of postulates for mathematics, either in whole or in part. The freedom of choice in the construction of mathematics, once the fundamental intuition is recognized, leaves the way open for setting up various postulate systems for any part of mathematics or for the whole science. This seems indeed to be the most reasonable attitude towards postulational theory. Each system of postulates for a particular field of knowledge is to be looked upon as a set of pronouncements in terms of undefined ideas, which are verifiable in that particular field, if the undefined ideas are suitably particularized. In the measure in which these postulates are independent they enlarge our knowledge of the structure of the field; in the measure in which they are non-categorical, they establish relations with other fields.

In the second place, even though constructed without any direct interplay of experienced reality, mathematics is not without value for practical life. Because through the agency of mathematical building, phenomena are linked together in causal sequences, which enable man to control the external world. "The conduct of man aims to observe as many as possible of these mathematical sequences, in order that, whenever an earlier element in such a sequence offers in actuality a better opportunity for taking hold of the situation than a later element in the same sequence, even though only the later one appeals to his instincts, he may choose the earlier one as the object of his acts." The mathematical universe thus becomes an accompaniment of the phenomenal universe, which assists man in his control of the latter.

3. *Mathematics and Mathematical Language.* The relation of Brouwer's thought to the Platonic point of view, hinted at above, is brought out once more in his insistent separation of mathematics from the language of mathematics. In Book VII of *The Republic*, Socrates is made to say, that "on one point at any rate we shall encounter no opposition from those who are even slightly acquainted with geometry, when we assert that this science holds a position which flatly contradicts the language employed by those who handle it." In quoting this passage and in commenting upon it, Boutroux gives more emphatic utterance to its thought: "It is known indeed, that the Platonists established a profound distinction between 'discourse' and 'intelligence', between written science, which is a didactic exposition of truths already known, and the *conception* of scientific truths, which is the direct product of our faculty of intuition in its dealing with the world of ideas."* For Brouwer, mathematics is a process of construction, and "of the mathematical building and reasoning, and in particular of the logical reasoning which men do within themselves, they try to evoke copies in other men by means of sounds and symbols, which also serve to aid their own memory." It seems that creative mathematicians cannot but receive with approval Brouwer's remark, that "in arguments concerning experiential realities, fitted into mathematical systems, logical principles are not the guide, but rather a regularity observed a posteriori in the accompanying language; and if one speaks in accordance with this regularity, but detached from mathematical systems, there is always a danger of paradoxes, like that of Epimenides." Mathematical proof without the use of words consists in establishing relations between different parts of the mathematical edifice, i. e. "when mathematical objects are given by means of their relations to elements or fragments of a mathematical edifice, one transforms these relations by a series of tautologies and thus one progresses step by step to the relations of the objects with other parts of the edifice."

* P. Boutroux, loc. cit., p. 35.

It is only in the language which accompanies this process for purposes of communication and memory, that logical forms arise. "The words of your mathematical demonstration are but the accompaniment of wordless mathematical building", Brouwer says to the logician, "and when you establish a contradiction, I simply observe that the construction cannot go on, that in the given edifice no room can be found for the posited structure. And when I make this observation I do not think of the Law of Contradiction." It is clear that the role of logic, as here conceived, is very different from the one usually attributed to it.

Before taking up more fully Brouwer's views of the relation of mathematics to logic, it will be of interest to insert the following passage: "The mathematical fact is independent of the logical or algebraic dress in which we seek to represent it; indeed, the idea which we have of it is richer and fuller than all the definitions which we can give of it, than all the forms or combinations of signs or of propositions by means of which we can express it. The expression of a mathematical fact is arbitrary, conventional. But the fact itself, that is to say, the truth which it contains, forces itself upon our mind apart from all conventions. Thus, one could not account for the development of mathematical theories, if one tried to consider the algebraic formulas and the logical combinations as the objects whose study the mathematician pursues. However, all the characteristics of these theories are easily explained, once one admits that the algebra and the logical propositions are but the language into which one *translates* a set of ideas and of objective facts."*

4. *Mathematics and Logic.* Indeed, Aristotelian logical reasoning is but a special kind of mathematical reasoning, namely that kind which is "concerned exclusively with relations of 'whole and part'." And the language which accompanies such logical reasoning is the *language of logical reasoning*, just as *mathematical language* is that which

* P. Boutroux, loc. cit., p. 203.

accompanies mathematical reasoning. Furthermore, these languages, themselves, like other parts of the phenomenal world can become the object of mathematical observation and study; thus arise *theoretical logic* as the mathematics of the language of logical reasoning, and *logistics* as the mathematics of the language of mathematical reasoning.

In view of these characterizations, it is not surprising to find little sympathy with the attempts to lay down logical foundations for mathematics. "A logical building up of mathematics, independent of the mathematical intuition is impossible — because in this way we obtain but a verbal edifice irrevocably apart from mathematics proper — and moreover a contradiction in terms, because a logical system, as well as mathematics itself, requires the fundamental intuition of mathematics."

The fundamental difference between the point of view of Brouwer concerning the nature of mathematics and that of Hilbert, as expressed in the latter's *Neubegründung*, referred to above, comes out clearly in the following sentences: "Suppose we have proved by some method or other, without having a mathematical interpretation in mind, that a logical system built up on the basis of some verbal axioms, is non-contradictory, i. e., that at no point of the development of the system two contradictory propositions will arise; and suppose that we then find a mathematical interpretation of the axioms, (which consists of requiring a construction of a mathematical edifice from elements which satisfy given mathematical relations). Does it then follow from the non-contradictoriness of the logical system that such a mathematical structure exists? No such thing has ever been proved by the postulationists" . . . "so, e. g., it has nowhere been proved, that if a finite number must satisfy a set of conditions which can be shown to be non-contradictory, that then this number actually exists."* If we compare this paragraph with Hilbert's system of undefined ideas and of postulates for mathematics, one is reminded of the phrase

* *Grondslagen*, p. 141.

of Poincaré, quoted elsewhere by Brouwer,* "Les hommes ne s'entendent pas, parce qu'ils ne parlent pas la même langue et qu'il y a des langues qui ne s'apprennent pas."

Concerning the logical paradoxes which have disturbed some mathematicians during the last twenty-five years and the attempts to resolve them by means of more refined logical methods, Brouwer holds that "they arise whenever regularity in the language which accompanies mathematics is extended so as to apply also in a language of mathematical words, which do not accompany mathematics." Moreover, "logistics concerns itself with mathematical language, instead of with mathematics, and consequently does not clarify mathematics; finally all paradoxes disappear if one restricts oneself to dealing with systems that are explicitly constructible on the basis of the fundamental intuition," i. e., if one gives priority to mathematics instead of to logic. This aspect of Brouwer's position again finds support elsewhere: "In other words, the most important advances which mathematicians make, are obtained not in perfecting the form, but in modifying the basis of the theory. These advances cannot be regarded as being of logical character."† . . . "In order to give mathematical theories a firm structure, we have decided to give them the form of logical systems; but, observing that these systems are artificial and can moreover be infinitely diversified, we realize that they neither constitute the whole of mathematics, nor its principal part. Behind the logical form there is something else."‡

But Brouwer does not merely indulge in a general criticism of the role of logic in mathematics; he proceeds to a discussion of the profoundly important question: "Is it allowable, in dealing with purely mathematical constructions and transformations temporarily to neglect the idea of the constructed mathematical system and to work with the accompanying verbal structure, guided by the principles of the *syllogism*,

* See this BULLETIN, vol. 20 (1913), p. 96.

† P. Boutroux, loc. cit., p. 168.

‡ Ibid. p. 170.

of *contradiction*, and of the *excluded middle*, confident that by evoking temporarily the idea of the reasoned mathematical constructions, every part of the argument could be validated in turn?"*

It should be clear that for the actual work of mathematical research this question, once one adopts Brouwer's conception of the character of mathematical thinking, is of primary importance. And his answer is *Yes*, as concerns the principles of the syllogism and of contradiction, but *No* for the law of the excluded middle. While the law of contradiction asserts that it is impossible for a proposition to be both true and false, the law of the excluded middle (L. E. M.) says that every proposition is either true or false. Its acceptance leads therefore to a belief in the solvability of every mathematical problem.† For, from Brouwer's point of view, this principle asserts that "for every hypostatized fitting into each other of systems in a definite way, either the actual construction can be made, or an insurmountable obstruction can be erected." If the proposition deals with fragments of a finite, definite, discrete system, this possibility will readily be granted, so that the L. E. M. may be considered as valid in dealing with such cases. For instance, of two positive integers, it can be affirmed that either they are relatively prime, or they possess a common divisor different from unity.

But the situation becomes different when we are dealing with infinite systems. Propositions concerning infinite systems can be dealt with systematically only when the use of complete induction is possible; in such a case the infinite system can be fitted in by the use of properties of an *arbitrary element*. On the other hand, the totality of the mathematical properties and contradictions derivable by means of complete induction forms what Brouwer calls a "denumerably un-

* A statement of the laws of thought will be found in any text on formal logic; see, e. g. Jevons, *Elementary Lessons in Logic*, p. 117.

† Compare Hilbert, *Mathematische Probleme*, GÖTTINGER NACHRICHTEN, 1900.

finished set," i. e., a set of which nothing but a denumerable subset can ever be explicitly exhibited, and such that whenever a denumerable subset is given, a new element of the set can always be derived from it by means of a previously defined process.* The possibility of systematically establishing the truth or falsity of a proposition concerning an arbitrarily proposed infinite system depends therefore upon finding among the denumerably unfinished set of mathematical properties and contradictions one which (eventually by means of complete induction, i. e. "by means of an element invariant over a denumerably infinite sequence") enables us to put the proposition as one which can be dealt with and decided, one way or the other, by the use of complete induction.

But the search for such a structure of property or contradiction cannot be carried out systematically; hence its success depends more or less upon good luck and cannot be assured a priori. Hence it is uncertain whether for an arbitrary proposition concerning a given infinite system either the construction or the obstruction can be established, and hence it is equally uncertain whether the L. E. M. is valid in such a case. But, still further, unjustified assumption that one or the other must be possible can never be detected; for that would mean that both the hypothesis of construction and that of obstruction would lead to an obstruction in the further process of construction, which conflicts with the law of contradiction.

It is on the basis of these considerations that Brouwer denies unlimited validity to the L. E. M., and that he reaches the following conclusion: "In mathematics, it is not certain whether or not all logic is permissible, and it is not certain whether it can be decided, whether or not all logic is permissible."

THE UNIVERSITY OF WISCONSIN

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* This notion, of which the set of well-ordered ordinals is an example, plays an important part in much of Brouwer's work.