NOTE ON STABILITY À LA POISSON*

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In this note will be developed certain consequences of a theorem due to Poincaré concerning "Stabilité à la Poisson".†

Suppose given a differential system

$$(1) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2 \cdots x_n), \qquad (i = 1, 2, \ldots, n),$$

such that the functions X_i are analytic in their arguments within an ordinary‡ closed region D, and suppose the equations (1) admit a positive multiplier M within this region. Then

$$(2) \quad \frac{\partial}{\partial x_1}(MX_1) + \frac{\partial}{\partial x_2}(MX_2) + \cdots + \frac{\partial}{\partial x_n}(MX_n) = 0.$$

In addition one can adopt either of the following hypotheses: (α) every solution of (1) which takes on a system of values $(x_1^0, x_2^0, \ldots, x_n^0)$ belonging to D at the time $t = t_0$ remains within or on the boundary of D for all values of t; (β) every solution of (1) which takes on a system of values $(x_1^0, x_2^0, \ldots, x_n^0)$ at $t = t_0$, belonging to an ordinary closed region D' interior to D remains within or on the boundary of D for all values of t.

In the discussion given by Poincaré, hypothesis (α) is made explicitly; but an examination of the argument shows that the same method of reasoning can be applied under hypothesis (β), to the solutions of (1) which, at $t = t_0$, take on values belonging to D'. The result may be stated as follows:

If $P_0(x_1^0, \ldots, x_n^0)$ is any interior point of D', Δ any ordinary closed region containing P_0 and interior to D',

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† *Les Méthodes Nouvelles de la Mécanique Céleste,* vol. 3, chapter 26.

‡ By "ordinary closed region" we mean an n-dimensional region bounded by an ordinary $(n-1)$-dimensional surface, and possessing a certain volume different from zero.

then there exists at least one point $P'(x_1', x_2', \ldots, x_n')$ belonging to Δ such that, given t_0 and $T > t_0$, there exists a value $\tau > T$ such that the solution of (1) taking on the values $(x_1', \ldots, x_n')$ at $t = t_0$ will take on values $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ belonging to Δ at the time $t = \tau$.

An application will be made to the case in which $X_i(0, 0, \ldots, 0) = 0$, $(i = 1, 2, \ldots, n)$, and in which the system (1) is known to possess *ordinary stability* in a certain neighborhood $|x_i| < A$. Ordinary stability may be said to exist in the given region under the following conditions: given any region (b): $|x_i| \leqq b$, $b \leqq A$, there exists a region (c): $|x_i| \leqq c$, $c \leqq b$, such that any solution of (1) which takes on a system of values belonging to (c) at $t = t_0$ lies within (b) for all values of t.

Suppose $c = C$ the value corresponding to $b = A$; the assumption of ordinary stability implies hypothesis (β) in which $D' = (C)$, $D = (A)$. Consequently there is stability à la Poisson within the region (C) if equations (1) admit a positive multiplier M within the region (A). This condition will be satisfied if equations (1) are in the canonical form

$$(3) \qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad (i = 1, 2, \ldots, m),$$

since equations (2) are satisfied by $M = 1$.

From an examination of the chapter of *Les Méthodes Nouvelles* mentioned above it can easily be seen that the assumptions concerning the functions X_i can be generalized. It is sufficient that the solutions of (1) be continuous functions of the initial values for $t = t_0$, and that they possess continuous partial derivatives of the first order with respect to them. These conditions will be satisfied if the partial derivatives $\partial X_i/\partial x_k$ exist and are continuous within and on the boundary of the region D considered,* $(i, k = 1, 2, \ldots, m)$.

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* Goursat, *Cours d'Analyse,* vol. 3, 2d edition, p. 13.